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CONTENTS

1

The Reduction of Certain Abelian Integrals with Application to the Summation of Infinite Series of Legendre's Functions

BY

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1 The object of the present note is to reduce certain abelian integrals to elliptic integrals The method of reduction followed here is the direct change of variables I have made an application of some of these reducible abelian integrals to the summation of certain series of Legendre's functions of non-integral parameters in terms of elliptic integrals

I wish to express my thanks to Professor Ganesh Prasad for the keen interest he has taken in this paper.

2 Consider the reduction of

$$\int \frac{dx}{(x^2-4g_2)^{\frac{1}{2}}(x^3-3g_2x-g_3)^{\frac{1}{m}}}. \tag{1}$$

Change the variable in this integral by means of the relation

$$x^3-3g_2x-g_3=z^m$$

Then from the Cardan's solution of the cubic,

$$x=p^{\frac{1}{3}}+q^{\frac{1}{3}}$$

where

$$p=\tfrac{1}{2}\{g_3+z^m+\sqrt{(g_3+z^m)^2-4g_2^3}\}$$

and

$$q=\tfrac{1}{2}\{g_3+z^m-\sqrt{(g_3+z^m)^2-4g_2^3}\}$$

If we put

$$u=\sqrt{g_3+z^m+2g_2^{\frac{3}{2}}}+\sqrt{g_3+z^m-2g_2^{\frac{3}{2}}}$$

$$v=\sqrt{g_3+z^m+2g_2^{\frac{3}{2}}}-\sqrt{g_3+z^m-2g_2^{\frac{3}{2}}}$$

and therefore

$$du=\frac{mz^{m-1}dz}{2\sqrt{g_3+z^m+2g_2^{\frac{3}{2}}}}+\frac{mz^{m-1}dz}{2\sqrt{g_3+z^m-2g_2^{\frac{3}{2}}}}$$

$$=\frac{mz^{m-1}dz\ \ u}{2\sqrt{(g_3+z^m)^2-4g_2^3}},$$

$$dv=\frac{-mz^{m-1}dz\ \ v}{2\sqrt{(g_3+z^m)-4g_2^3}},$$

we get

$$x=p^{\frac{1}{3}}+q^{\frac{1}{3}}$$

$$=\left(\frac{u}{2}\right)^{\frac{2}{3}}+\left(\frac{v}{2}\right)^{\frac{2}{3}}$$

$$=\frac{1}{2^{\frac{2}{3}}}[u^{\frac{2}{3}}+v^{\frac{2}{3}}]$$

and

$$dx=\frac{1}{2^{\frac{2}{3}}}\,\frac{2}{3}\left\{\frac{du}{u^{\frac{1}{3}}}+\frac{dv}{v^{\frac{1}{3}}}\right\}$$

$$=\frac{2^{\frac{1}{3}}}{3}\,\frac{mz^{m-1}dz}{\sqrt{(g_3+z^m)^2-4g_2^3}}\,(u^{\frac{2}{3}}-v^{\frac{2}{3}})$$

Now

$$uv=4g_2^{\frac{3}{2}}$$

therefore

$$(x^2-4g_2)^{\frac{1}{2}}=\frac{1}{2^{\frac{2}{3}}}(u^{\frac{2}{3}}-v^{\frac{2}{3}})$$

$$\int \frac{dx}{(x^2-4g_2)^{\frac{1}{2}}(x^3-3g_2x-g_3)^{\frac{1}{m}}} \text{ becomes equal to}$$

$$\frac{m}{3}\int \frac{z^{m-2}dz}{\sqrt{(g_3+z^m)^2-4g_2^3}}$$

$$= \frac{m}{3}\int \frac{z^{m-2}dz}{\sqrt{z^{2m}+az^m+b}} \qquad (2)$$

where $a=2g_3$ and $b=g_3^2-4g_2^3$

(2) is reducible to elliptic integrals when $m=3$, 4, 6, the last case, *viz* , of $m=6$ having been considered in my previous paper The case $m=2$ of (1) is well known as was first reduced by Hermite

3 Next take the abelian integral

$$\int \frac{d\phi}{(1-k^2 \sin^2\phi)^{\frac{m}{6}}}$$

where m is 1 or 5 and $k=\cos a$

Putting $1-\cos^2 a \sin^2\phi=x^6 \sin a$, we get the integral reduced to

$$-3(\sin a)^{\frac{1}{2}-\frac{m}{6}}\int \frac{x^{5-m}dx}{\sqrt{(x^6-\sin a)(1-x^6 \sin a)}}$$

The reduction of this integral to elliptic integral is known* when $m=5$, and

$$\int \frac{d\phi}{(1-\cos^2 a \sin^2\phi)^{\frac{5}{6}}} = \frac{3i}{2}(\sin a)^{-\frac{5}{6}}\left\{\frac{1}{\sqrt{6}}\wp^{-1}\left(\frac{y}{2}, 3, -\frac{c}{2}\right)\right.$$

$$\left.-\frac{1}{\sqrt{2}}\wp^{-1}\left(z, 3, \frac{c}{2}\right)\right\},$$

* See my paper in Vol 25 (pp 159-166) of this *Bulletin* I take this oppartunity to mention the fact that in his paper, entitled "On the reduction of certain Abelian Integrals by Trigonometrical Substitution," Mr B Dayal obtains the result for $m=1$ by means of a trigonometrical transformation This fact has been brought to my notice by Prof G Prasad, Mr Dayal's paper will appear soon in the *Proc. B. M. S.*, Vol 15

when $m=1$

$$\int \frac{d\phi}{(1-\cos^2\alpha \sin^2\phi)^{\frac{1}{6}}}$$

$$=\frac{3i}{2}(\sin\alpha)^{-\frac{1}{6}}\left\{\frac{1}{\sqrt{2}}\mathfrak{C}^{-1}\left(z, 3, \frac{c}{2}\right)+\frac{1}{\sqrt{6}}\mathfrak{C}^{-1}\left(\frac{y}{2}, 3-\frac{c}{2}\right)\right\},$$

where $x^2+\frac{1}{x^2}=2z$ and $3y=\frac{2z^3-c}{z^2-1}$, c being $\sin\alpha+\operatorname{cosec}\alpha$.

4 Let us make an application of this reduction to the summation of certain series.

The series

$$P_{-\frac{5}{6}}(\cosh\sigma)+\tfrac{1}{6}rP_{\frac{1}{6}}(\cosh\sigma)+\frac{1}{6}\frac{7}{12}r^2P_{\frac{7}{6}}(\cosh\sigma)+\frac{1}{6}\frac{7}{12}\frac{13}{18}r^3P_{\frac{13}{6}}(\cosh\sigma)+\ldots$$

has a sum

$$\frac{2}{\pi(e^{\sigma}-r)^{\frac{1}{6}}}\int_0^{\frac{\pi}{2}}\frac{d\phi}{(1-k^2\sin^2\phi)^{\frac{1}{6}}},$$

k^2 standing for $\frac{2\sinh\sigma}{e^{\sigma}-r}$ and $r<e^{-\sigma}$

Now $$\int_0^{\frac{\pi}{2}}\frac{d\phi}{(1-\cos^2\alpha\sin^2\phi)^{\frac{1}{6}}}$$

$$=-3(\sin\alpha)^{\frac{1}{3}}\int_{(\operatorname{cosec}\alpha)^{\frac{1}{6}}}^{(\sin\alpha)^{\frac{1}{6}}}\frac{x^4\,dx}{\sqrt{(x^6-\sin\alpha)(1-x^6\sin\alpha)}}$$

$$=-3(\sin\alpha)^{\frac{1}{3}}\int_1^{(\sin\alpha)^{\frac{1}{6}}}\frac{(1+x^4)\,dx}{\sqrt{(x^6-\sin\alpha)(1-x^6\sin\alpha)}}$$

$$=\frac{3i(\sin\alpha)^{-\frac{1}{6}}}{\sqrt{6}}\mathfrak{C}^{-1}\left(\frac{y}{2}, 3, -\frac{c}{2}\right)$$

Therefore the sum of the series

$$P_{-\frac{5}{6}}(\cosh\sigma)+\tfrac{1}{6}rP_{\frac{1}{6}}(\cosh\sigma)+\frac{1\ 7}{6\ 12}\ r^2P_{\frac{7}{6}}(\cosh\sigma)+\ \ldots$$

$$=\frac{i\sqrt{6}\ (\sin\alpha)^{-\frac{1}{6}}}{\pi\ (e^{\sigma}-r)^{\frac{1}{6}}}\mathfrak{C}^{-1}\left(\frac{y}{2},3,-\frac{c}{2}\right)$$

Similarly the sum of the series

$$P_{-\frac{1}{6}}(\cosh\sigma)+\tfrac{5}{6}rP_{\frac{5}{6}}(\cosh\sigma)+\frac{5\ 11}{6.\ 12}\ r^2P_{\frac{11}{6}}(\cosh\sigma)+\ldots$$

is
$$\frac{\sqrt{6}i}{\pi}\ \frac{(\sin\ \alpha)^{-\frac{5}{6}}}{(e^{\sigma}-r)^{\frac{5}{6}}}\ \mathfrak{C}^{-1}\left(\frac{y}{2},3,-\frac{c}{2}\right)$$

The two series seem to be related The relation becomes evident from the following considerations

$$P_{n-\frac{1}{2}}(\cosh\sigma)=\frac{1}{\pi}\int_0^{\pi}\frac{d\phi}{(\cosh\sigma+\sinh\sigma\cos\phi)^{n+\frac{1}{2}}}$$

$$=\frac{2}{\pi}\ \frac{1}{e^{(\sigma n+\frac{1}{2})}}\int_0^{\frac{\pi}{2}}\frac{d\theta}{(1-k^2\sin^2\theta)^{n+\frac{1}{2}}}$$

where
$$k^2=\frac{2\sinh\sigma}{e^{\sigma}}\ \text{and}\ \theta=\frac{\phi}{2}.$$

Thus
$$\frac{2}{\pi}\cdot\ \frac{1}{e^{\frac{\sigma}{6}}}\int_0^{\frac{\pi}{2}}\frac{d\theta}{(1-k^2\ \sin^2\theta)^{\frac{1}{6}}}=P_{-\frac{5}{6}}(\cosh\sigma)$$

and
$$\frac{2}{\pi}\ \frac{1}{e^{\frac{5\sigma}{6}}}\int_0^{\frac{\pi}{2}}\frac{d\theta}{(1-k^2\ \sin^2\theta)^{\frac{5}{6}}}=P_{-\frac{1}{6}}(\cosh\sigma)$$

From the properties of Legendre's functions,

$$P_{-\frac{1}{6}}=P_{-\frac{5}{6}}.$$

Hence if we take a new quantity σ_1 such that

$$\frac{\sinh\sigma_1}{e^{\sigma_1}}=\frac{\sinh\sigma}{e^{\sigma}-r}$$

so that $$e^{-2\sigma_1}=\frac{e^{-\sigma}-r}{e^{\sigma}-r}$$

$$P_{-\frac{5}{6}}(\cosh\sigma)+\tfrac{1}{6}rP_{\frac{1}{6}}(\cosh\sigma)+\frac{1}{6}\frac{7}{12}\,r^2P_{\frac{7}{6}}(\cosh\sigma)+$$

$$=\frac{\sqrt{6}i}{\pi}\frac{1}{(1-2r\cosh\sigma+r^2)^{\frac{1}{12}}}\mathfrak{C}^{-1}\left(\frac{y}{2},3,-\frac{c}{2}\right)$$

$$=\frac{1}{(1-2r\cosh\sigma+r^2)^{\frac{1}{12}}}P_{-\frac{5}{6}}(\cosh\sigma_1)$$

and

$$P_{-\frac{1}{6}}(\cosh\sigma)+\tfrac{5}{6}rP_{+\frac{5}{6}}(\cosh\sigma)+\frac{5}{6}\frac{11}{12}\,r^2P_{\frac{11}{6}}(\cosh\sigma)+\ldots$$

$$=\frac{\sqrt{6}i}{\pi}\frac{1}{(1-2r\cosh\sigma+r^2)^{\frac{5}{12}}}\mathfrak{C}^{-1}\left(\frac{y}{2},3,-\frac{c}{2}\right)$$

$$=\frac{1}{(1-2r\cosh\sigma+r^2)^{\frac{5}{12}}}P_{-\frac{1}{6}}(\cosh\sigma_1)$$

$$\cosh\sigma_1=\frac{\cosh\sigma-r}{(1-2r\cosh\sigma+r^2)^{\frac{1}{2}}}.$$

Similarly the sums of the following series can be found —

$$Q_{-\frac{5}{6}}(\cosh\sigma)+\tfrac{1}{6}rQ_{\frac{1}{6}}(\cosh\sigma)+\frac{1}{6}\frac{7}{12}\,r^2Q_{\frac{7}{6}}(\cosh\sigma)+\ldots$$

$$=\frac{1}{(1-2r\cosh\sigma+r^2)^{\frac{1}{12}}}Q_{-\frac{5}{6}}(\cosh\sigma_1)$$

and

$$Q_{-\frac{1}{6}}(\cosh\sigma)+\tfrac{5}{6}rQ_{\frac{5}{6}}(\cosh\sigma)+\frac{5.11}{6\ 12}\,r^2Q_{\frac{11}{6}}(\cosh\sigma)+\ldots$$

$$=\frac{1}{(1-2r\cosh\sigma+r^2)^{\frac{5}{2}}}Q_{-\frac{1}{6}}(\cosh\sigma_1)$$

where $\cosh\sigma_1=\dfrac{\cosh\sigma-r}{(1-2r\cosh\sigma+r^2)^{\frac{1}{2}}}$

Lastly we shall find the sum of the series

$$Q_{\frac{1}{2}}^{-\frac{1}{3}}(\cosh\sigma)-\tfrac{1}{3}\,Q_{\frac{5}{2}}^{-\frac{1}{3}}(\cosh\sigma)+\tfrac{1}{5}\,Q_{\frac{9}{2}}^{-\frac{1}{3}}(\cosh\sigma)-$$

It is known* that

$$Q^{m}_{n-\frac{1}{2}}(\cosh\sigma)=(-1)^{m}\,\frac{2^{m}\Pi(m-\frac{1}{2})\Pi(-\frac{1}{2})}{\pi}\int_0^{\pi}\frac{\sinh^{m}\sigma\cos n\phi\,d\phi}{\{2(\cosh\sigma-\cos\phi)\}^{m+\frac{1}{2}}}$$

Let S denote the sum of the series

$$S=(-1)^{-\frac{1}{3}}\frac{\Pi(-\frac{5}{6})\,\Pi(-\frac{1}{2})}{\sqrt{2}\pi}\,(\sinh\sigma)^{-\frac{1}{3}}$$

$$\int_0^{\pi}\frac{d\phi}{(\cosh\sigma-\cos\phi)^{\frac{1}{6}}}\left\{\sum_{p=1}^{\infty}(-1)^{p-1}\frac{\cos(2p-1)\phi}{2p-1}\right\}$$

Now $\displaystyle\sum_{p=1}^{\infty}(-1)^{k-1}\frac{\cos(2p-1)\phi}{2p-1}=\frac{\pi}{4}$ as $\cos\phi$ is positive

$$=-\frac{\pi}{4}\text{ as }\cos\phi\text{ is negative}$$

Therefore

$$S=(-1)^{-\frac{1}{3}}\frac{\Pi(-\frac{5}{6})\Pi(-\frac{1}{2})}{4\sqrt{2}}\left[\int_0^{\frac{\pi}{2}}\frac{d\phi}{(\cosh\sigma-\cos\phi)^{\frac{1}{6}}}\right.$$

$$\left.-\int_0^{\frac{\pi}{2}}\frac{d\theta}{(\cosh\sigma+\cos\theta)^{\frac{1}{6}}}\right]$$

where $\phi=\pi-\theta$

But

$$\int_0^{\frac{\pi}{2}}\frac{d\theta}{(\cosh\sigma+\cos\theta)^{\frac{1}{6}}}=\frac{2}{\left(2\cosh^{2}\frac{\sigma}{2}\right)^{\frac{1}{6}}}\int_0^{\frac{\pi}{4}}\frac{d\lambda}{(1-k^{2}\sin^{2}\lambda)^{\frac{1}{6}}}$$

where $\lambda=\frac{\theta}{2}$, and $k^{2}=\operatorname{sech}^{2}\frac{\sigma}{2}$.

* E. W. Hobson, Phil. Trans., A, Vol 187, 1896.

Also

$$\int_0^{\frac{\pi}{2}} \frac{d\phi}{(\cosh\sigma-\cos\phi)^{\frac{1}{6}}}=\int_0^{\frac{\pi}{2}} \frac{d\phi}{\left(\cosh\sigma-1+2\sin^2\frac{\phi}{2}\right)^{\frac{1}{6}}}$$

$$=\frac{2}{\left(2\cosh^2\frac{\sigma}{2}\right)^{\frac{1}{6}}}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{d\psi}{(1-k^2\sin^2\psi)^{\frac{1}{6}}}$$

where $k^2=\operatorname{sech}^2\frac{\sigma}{2}$ and $\sin\frac{\phi}{2}=\cos\psi$.

Therefore

$$S=(-1)^{-\frac{1}{3}}\frac{\Pi(-\frac{5}{6})\Pi(-\frac{1}{2})}{2\sqrt{2}}\;\frac{(\sinh\sigma)^{-\frac{1}{3}}}{\left(2\cosh^2\frac{\sigma}{2}\right)^{\frac{1}{6}}}\left\{\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{d\psi}{(1-k^2\sin^2\psi)^{\frac{1}{6}}}\right.$$

$$\left.-\int_0^{\frac{\pi}{4}}\frac{d\lambda}{(1-k^2\sin^2\lambda)^{\frac{1}{6}}}\right\}$$

Thus S can be expressed as a sum of elliptic integrals After some easy calculations we get

$$S=(-1)^{-\frac{1}{3}}\frac{\Pi(-\frac{5}{6})\,\Pi(-\frac{1}{2})\mathfrak{Z}}{4\sqrt{\sinh\sigma}}\left\{\wp^{-1}\left(z,3,\frac{c}{2}\right)+\frac{1}{\sqrt{3}}\,\wp^{-1}\left(y,3,-\frac{c}{2}\right)\right\}$$

where z and y are expressible in term of σ, though the actual expressions are very complicated

Exactly in a similar manner we can show that the sum of the series

$$Q^{\frac{1}{3}}_{\frac{1}{2}}(\cosh\sigma)-\tfrac{1}{3}\,Q^{\frac{1}{3}}_{\frac{5}{2}}(\cosh\sigma)+\tfrac{1}{6}\,Q^{\frac{1}{3}}_{\frac{9}{2}}(\cosh\sigma)\qquad .\qquad \ldots$$

is

$$(-1)^{\frac{1}{3}}\mathfrak{Z}\,\frac{\Pi(-\frac{1}{6})\,\Pi(-\frac{1}{2})}{4\sqrt{\sinh\sigma}}\left\{\frac{1}{\sqrt{3}}\wp^{-1}\left(\frac{y}{2},3,-\frac{c}{2}\right)-\wp^{-1}\left(z,3,\frac{c}{2}\right)\right\}$$

Bull. Cal Math. Soc , Vol. XXVI, No 1 (1934).

2

"On some results involving Bessel Functions"

N G Shabde

(*Edinburgh University*)

1 The object of the present note is to obtain some integrals and series involving Bessel functions The results on account of their general elegance seem to be worthy of being placed on record The results appear to be new

2 We take Bateman's expansion*

$$(2\,1) \qquad x\, J_0(\sqrt{u}\; x) = \sum_{n=0}^{\infty} (4n+2) P_n(1-2\mu) J_{2n+1}(x)$$

which is valid and uniformly convergent for $0 \leq u \leq 1$ This gives for $-1 < \mu < 1$

$$(2.2) \qquad \int_0^1 x\, J_0(\sqrt{u}\cdot x)\, u^{\frac{\mu+1}{2}}\, (1-u)^{-\frac{\mu+1}{2}}\, du$$

$$= \sum_{n=0}^{\infty} (4n+2) J_{2n+1}(x) \int_0^1 P_n(1-2u) u^{\frac{\mu+1}{2}}\, (1-u)^{-\frac{\mu+1}{2}}\, du$$

$$= \pi \sec \frac{\pi\mu}{2} \sum_{n=0}^{\infty} (4n+2) J_{2n+1}(x)\;\; F_n(\mu)$$

* "On Expansion of an arbitrary function in a series of Bessel functions," *Mess of Math*, 36 (1907), 31-37

where $F_n(\mu)$ is a polynomial discussed recently by Bateman* and defined as

$$(1+t)-\mu.F\left(\frac{1-\mu}{2},\frac{1-\mu}{2}, 1, t^2\right)=\sum_{n=0}^{\infty} t^n.F_n(\mu), \ (t) < 1.$$

On putting $u=\sin^2\theta$ in the left hand side of (2 1) we have finally

$$\int_0^{\frac{\pi}{2}} J_0(x \sin\theta) \tan^{\mu}\theta \, d\theta=\frac{\pi}{2x} \sec\frac{\pi\mu}{2} \sum_{n=0}^{\infty} (4n+2) J_{2n+1}(x) F_n(\mu)$$

(2 3) $\qquad -1<\mu<1$

3 We have† again

(3 1) $\qquad r J_0(r\cos\theta\cos\omega) J_0(r\sin\theta\sin\omega)$

$$=\sum_{n=0}^{\infty} (-1)^n(4n+2)J_{2n+1}(r)P_n(\cos 2\theta)P_n(\cos 2\omega)$$

or

(3 2) $\qquad r J_0(r\cos\theta \sqrt{1-u}) J_0(r\sin\theta\sqrt{u})$

$$=\sum_{n=0}^{\infty} (-1)^n(4n+2)J_{2n+1}(r) \ P_n(\cos 2\theta)P_n(1-2u)$$

which is valid and uniformly convergent for $0\leq u\leq 1$

This gives when $-1<\mu<1$

(3 3) $\displaystyle\int_0^1 J_0 (r\cos\theta\sqrt{1-u}).J_0(r\sin\theta.\sqrt{u}) \, du \, u^{\frac{\mu+1}{2}} (1-u)^{-\frac{\mu+1}{2}}$

$$=\frac{1}{r} \sum_{n=0}^{\infty} (-1)^n(4n+2)J_{2n+1}(r)P_n(\cos 2\theta)$$

$$\int_0^1 P_n(1-2u)u^{\frac{\mu+1}{2}}(1-u)^{-\frac{\mu+1}{2}} . du$$

* "Some properties of a certain set of Polynomials," *Tohôku Math. Journal*, 37 (1933), 23 38.

† Bateman, "A generalisation of Legendre Polynomials," *Proc. London Math Soc*, (2), 3, (1905), 111-123, 116.

$$=\frac{1}{r\cos\frac{\pi\mu}{2}}\sum_{n=0}^{\infty}(-1)^n(4n+2)J_{2n+1}(r)P_n(\cos 2\theta)F_n(\mu)$$

Putting $u=\sin^2\phi$ on the left hand side of (3 3) we have

$$\int_0^{\frac{\pi}{2}} J_0(r\cos\theta\cos\phi)J_0(r\sin\theta\sin\phi)\tan^{\mu}\phi\, d\theta$$

$$=\frac{\pi}{2r\cos\frac{\pi\mu}{2}}\sum_{n=0}^{\infty}(-1)^n(4n+2)J_{2n+1}(r)P_n(\cos 2\theta)F_n(\mu)$$

4 A particular case of Sonine's * integrals gives

$$\int_0^{\infty} J_m(mx)\, x^q dx=\left(\frac{2}{m}\right)^q\frac{1}{m}\cdot\frac{\Pi\left(\frac{m+q-1}{2}\right)}{\Pi\left(\frac{m-q-1}{2}\right)},\quad q<\tfrac{1}{2}\text{ and}$$

$R\,(m+q+1)>0$

We take in (4 1) $m>0$ and $-1<q<\frac{1}{2}$

To (4 1) we apply the converse of a Fourier-Bessel Integral theorem given by MacRobert,†

that

(4 2) If $\quad f(\lambda)=\int_p^q\phi(\rho)J_\rho(\rho\lambda)\rho d\rho \qquad 0\leq p<q$ then

$$\int_0^{\infty}\left(\lambda=\frac{1}{\lambda}\right)f(\lambda)J_m(m\lambda)d\lambda=\left\{\tfrac{1}{2}\phi(m+0)+\phi(m-0)\right\},\ p<m<q,$$

$$0<m<p \text{ or } m>q$$

* *Math Annalen*, Bd 16

† "Fourier Integrals," *Proc Royal Soc. Edin*, 51 (1930-31), 121

this gives at once

$$(4.3)\quad \int_0^\infty \left(\frac{2}{\rho}\right)^q \frac{\Pi\left(\frac{\rho+q-1}{2}\right)}{\Pi\left(\frac{\rho-q-1}{2}\right)} J_\rho(\rho\lambda)\,d\rho = \frac{\lambda^{q+1}}{\lambda^2-1}, \quad \lambda \geq 0, \ -1 < q <$$

since in this case $\left(\lambda-\frac{1}{\lambda}\right)f(\lambda)=\lambda^q$ when $q=0$ this reduces to

$$(4.4)\quad \int_0^\infty J_\rho(\rho\lambda)\,d\rho = \frac{\lambda}{\lambda^2-1}, \quad \lambda \geq 1,$$

given by MacRobert in his paper (*loc cit*, p 123).

5 Taking the expansions

$$(5.1)\quad xJ_0\{x\cos\theta\sqrt{1-u^2}\}J_0\{x\sin\theta\, u\}$$

$$= \sum_{n=0}^{\infty} (-1)^n(4n+2)J_{2n+1}(x)P_n(\cos 2\theta)P_n(1-2u^2)$$

and

$$(5.2)\quad P_p(1-2u^2)P_q(1-2u^2)$$

$$= \sum_{r=0}^{q} \frac{A_{p-r}A_rA_{q-r}}{A_{p+q-r}} \left(\frac{2p+2q-4r+1}{2p+2q-2r+1}\right) P_{p+q-2r}(1-2u^2)$$

where $\quad 0 \leq u \leq 1,\ A_r = \frac{2^r(\frac{1}{2})_r}{r!},\ p \geq q$ and $A_{-r} = \frac{(-1)^r 2^r}{(\frac{1}{2})_r r!}$

and making use of the integral* by Mitra

$$(5.3)\quad \int_0^1 e^{-2uk}P_n(1-2u^2)\,du = I_{n+\frac{1}{2}}(k)\ K_{n+\frac{1}{2}}(k)$$

* "On certain integrals and expansions involving Bessel Functions," *Bull. Cal. Math Soc* 25 (1933), 81-98

we get

$$(5\ 4)\qquad \int_0^1 J_0(\sqrt{1-u^2}\ x\cos\theta)J_0(ux\sin\theta)e^{-2uk}du$$

$$=\frac{1}{x}\sum_{n=0}^{\infty}(-1)^n(4n+2)J_{2n+1}(x)P_n(\cos 2\theta)I_{n+\frac{1}{2}}(k)\ K_{n+\frac{1}{2}}(k)$$

and

$$(5\ 5)\qquad \int_0^1 P_p(1-2u^2)P_q(1-2u^2)e^{-2uk}du$$

$$=\sum_{r=0}^{q}\frac{A_{p-r}A_rA_{q-r}}{A_{p+q-r}}\left(\frac{2p+2q-4r+1}{2p+2q-2r+1}\right)I_{p+q-2r+\frac{1}{2}}K_{p+q-2r+\frac{1}{2}}(k)$$

6 Finally as a particular case of a result by Chaundy * we obtain an interesting integral involving Bessel functions

Chaundy's result runs as

$$(6\ 1)\qquad 2K_\mu(z)K_\nu(z)$$

$$=\int_0^\infty e^{-\frac{1}{2}(t+t^{-1})z}W_{\frac{1}{2},\frac{1}{2}(\mu-\nu)}(zt^{-1})K_{\frac{1}{2}(\mu+\nu)}(\tfrac{1}{2}zt)\frac{dt}{\sqrt{zt}}$$

$R(z)$ being greater than zero

Taking z to be real $(=x)$; $\mu=\nu+1$ and remembering that

$$W_{\frac{1}{2},\frac{1}{2}}\left(\frac{x}{t}\right)=\Gamma(\tfrac{3}{2})\frac{x}{t\pi}\left[K_0\left(\frac{x}{2t}\right)-K_1\left(\frac{x}{2t}\right)\right]$$

* *Quarterly Journal of Math*, 2 (1931), p 153, result (40) "On Products of Bessel's Functions," pp 144-154

we have

$$(6\,2)\quad 2K_{\nu+1}(x)K_{\nu}(x)$$

$$=\frac{\Gamma(\frac{3}{2})}{\pi}x\int_0^\infty \frac{e^{-\frac{1}{2}(t+t^{-1})x}}{t}\left[K_0\left(\frac{x}{2t}\right)-K_1\left(\frac{x}{2t}\right)\right]$$

$$\times K_{\nu+\frac{1}{2}}\left(\frac{xt}{2}\right)\frac{dt}{\sqrt{xt}}$$

Bull Cal Math Soc., Vol XXVI, No. 1 (1934).

3

ON A CERTAIN LEMMA WITH APPLICATIONS TO THE THEORY OF DERIVATES

BY

A. N SINGH

(*Lucknow*)

1 The derivates of functions have been studied from two different stand points In one case particular conditions have been imposed on the functions or on the derivates and the character of the derivates, especially as regards the existence of the differential coefficient has been studied Results of this type of study are some well known theorems, *e g.*, the theorem about the existence of the differential coefficient almost everywhere of a function of bounded variation given by Lebesgue* for the case of continuous functions and then generalised by W H Young† and Montel ‡ The generalisations of the mean value theorem given by W. H Young and G C Young§ and A N Singh,|| belong to this type of study On the other hand, attempts have been made to investigate the character of the sets of points at which the differential co-efficient does not exist, and at which there exist certain given relations among the values of the derivates ¶ The study of the metric character of such sets has

* Lebesgue, *Leçons sur l'intégration* (1904), p 128

† W H Young, *Quarterly Journal of Mathematics*, Vol XLII (1911), p 79.

‡ *Comptes Rendus*, Vol CLV (1912), p 1478

§ W H Young and G C Young, *Quarterly Journal of Math* , Vol XL, p 10

|| A N Singh, *Bull Cal Math Soc* , Vol XIX (1928), pp 43-49

¶ A study of the character of the *sets of continuous functions* possessing given peculiarities as regards their derivates has been recently made by Banach [*Studia Math* , Vol III (1931), pp 174 179], Mazurkiewicz [*Studia Math* , Vol III (1931), pp 92-94], Saks [*Fundamenta Math* , XIX (1932), pp 211-219] and Jarnik [*Fundamenta Math* , XXI (1933), pp 48-58]

reached a degree of finality and the general result established may be enunciated as follows *If $f(x)$ be any function, finite at each point, and if a set of measure zero be left out of account, then the derivates at the other points may have one or more of the following characteristics*

(1) *they are finite and equal,*

(2) *the two upper derivates are $+\infty$ and the two lower derivates are $-\infty$,*

(3) *the upper derivate on one side is $+\infty$, the lower derivate on the other side is $-\infty$, and the other two derivates are finite and equal*

The above theorem was first proved for the case of continuous functions by A Denjoy * It was extended to the case of measurable functions by G C Young † Some of the above results have been shown to hold for all functions measurable or not by M Banach‡ and Saks § There are, on the other hand, some results given by W H Young which relate to the descriptive character of the set at which the derivates of a function possess a given peculiarity For example, we have the theorem :

The points at which one at least of the derivates of a given function is infinite form an ordinary inner limiting set ||

The present paper gives an investigation of the derivates of a function along the above lines In this paper I restrict myself to absolutely continuous functions and to functions of bounded variation, and investigate the properties of the derivates of such functions in relation to sets of the second category Of the results obtained about absolutely continuous functions may be mentioned the following .

If $f(x)$ be an absolutely continuous function, and if one of the four derivates of $f(x)$ be known to be zero at each point of a set CG complementary to a set G of the first category in (a, b), then $f(x)$ is a constant throughout (a, b)

* *Journal de Math* (7), t I (1915), p 105

† *Proc Lond Math Soc.* (2), Vol XV (1916), p 360

‡ *Comptes Rendus*, t 168 , see also Rajchman and Saks, *Fundamenta Mathematicae*, t IV (1923), pp 204-213

§ *Fund Math* , t V (1924), pp. 98 104

|| *Arkiv för Matematik, Astr och Fysik*, Vol 1 (1903) , see also Brodén, *Acta univ Lund*, Vol XXXIII (1897), p 31

It is well known that the similar theorem when $f(x)$ is taken to be continuous requires one of the derivates to be zero everywhere except at an enumerable set * The above theorem can be applied at once to prove the equality except for an additive constant of two primitive functions if they have the same derivates everywhere except at a set of the first category A similar result regarding functions of bounded variation may be given Of the various results relating to the derivates of functions of bounded variation may be mentioned the following

If $f(x)$ be of bounded variation in (a, b), then one of the four derivates of $f(x)$ can not be greater than $\mathrm{P}_a^b/(b-a)$ everywhere in (a, b) exception being made of a set G of the first category in (a, b), at which nothing is known as regards the values of the derivates, P_a^b being the total positive variation of $f(x)$ in (a, b)

A corollary of the above result is

One of the four derivates of a function of bounded variation can not be infinite at all points of a set CG which is complimentary to a set G of the first category in (a, b)

According to the theorem of Young quoted above, the set where one of the derivates is infinite is an ordinary inner limiting set and is therefore of the second category if it is everywhere dense The above corollary shows that this set can not be the compliment of a set of the first category *This result can be utilised to divide an interval (a, b) into two everywhere dense sets of the second category* †

Another interesting result regarding functions of bounded variation is

Every function of bounded variation possesses finite derivates at each point of a set of the second category

All the results of this paper are believed to be new They depend on the following lemma

If with each point of a set CG in (a, b) there is given one interval Δ with that point as left (right) end point, and with each point of the complimentary set G all intervals δ tending to zero (in length) with that point as left (right) end point, then, provided that G is a set of the first category

* See Hobson, *Theory of Functions, etc*, Vol I (1927), p 365

† Mahlo (*Leipz Ber*, Vol LXV (1913), p 283) has developed another method by which such a division can be made

in (a, b), there exists a chain of Δ and δ intervals reaching from a to b and such that

$$m_e \Delta \geq (b-a)-\eta,$$

where η is any arbitrarily small positive number

Two proofs of this lemma have been given The first proof establishes the possibility of giving a method by which a chain of Δ and δ intervals may be constructed in (a, b) The second proof gives a law of construction which applied successively will lead to the required chain of Δ and δ intervals It is believed that the chain is not unique and that different methods of construction would give different chains

It would be interesting to obtain independent proofs of the theorems given in this paper without the use of Lemma II

2 *Lemma* I *If CG be a set complimentary to a set of the first category G, it is possible to determine uniquely a point of CG in any interval with a point of G as left end point*

As the set G is a set of the first category, it is the outer limit of a sequence of non-dense closed sets each of which contains the preceding one Let the sequence which defines G be $g_1, g_2, g_3,$ $g_n,$ As the sets are non-dense, let $\Delta_1, \Delta_2, \Delta_3,$. $\Delta_n,$ be the sets of intervals complementary to g_1, g_2, g_3 ... . g_n . respectively

Now, let x be a point of G. Consider the interval $(x, x+h)$ The point x may be either a limiting point on the right of the end points of the Δ intervals or is not such a limiting point but is an end point

In case it is not a limiting point (on the right) then exists a greatest interval $(x, x+\eta)$ which does not contain any point of G The point $x+\frac{1}{2}\eta$ is a point of CG and is uniquely determined

In case it is a limiting point of end points, a point of CG may be determined as follows:

Let r_1 be the lowest value of n such that one or more intervals of the set Δr_1 are contained in the open interval $(x, x+h)$ Of the several intervals of Δr_1 contained in $(x, x+h)$ one or a finite number are the greatest Of these greatest intervals of Δr_1, select that which is nearest to $x+h$, let it be δr_1. We can similarly determine an interval δr_2 contained in δr_1, $(r_2 > r_1)$ and so on Continuing in

this manner we obtain an uniquely defined sequence of intervals one within the other

$$\delta r_1, \delta r_2, \delta r_3, \quad \ldots \quad \delta r_n, \quad \ldots$$

This sequence of intervals defines uniquely either a single point ξ or a single interval (ξ_1, ξ_2) In the first case the point ξ is the required point of CG, and in the second case the point $\frac{\xi_1+\xi_2}{2}$ may be taken as the required point of CG The point ξ is evidently not a point of G as it is interior to intervals complementary to G For the same reason $\frac{\xi_1+\xi_2}{2}$ is not a point of G

3 *Lemma II If with each point of a set CG in (a, b) there is given one interval Δ with that point as left (right) end point, and with each point of the complementary set G all intervals δ tending to zero (in length) with that point as left (right) end point, then, provided that G is a set of the first category in (a, b), there exists a chain of Δ and δ intervals reaching from a to b, and such that*

$$m_e \Delta \geq (b-a) - \eta$$

where η is any arbitrary small positive number

(3 1) We may without loss of generality suppose that a is a point of CG, then with a as left end point there is a Δ interval, say Δ_1 If the right hand end point of Δ_1 is a point of CG, then there is another Δ interval, say Δ_2, abutting on Δ_1 If the right end point of Δ_1 is a point of G (say g_1), then a point of CG can, by Lemma I, be determined in the interval $(g_1, g_1+\frac{1}{2}\epsilon\Delta_1)$ Call this point c_2. Then with c_2 as left end point we have a Δ interval, say Δ_2. We have thus two cases

(1) Δ_1 and Δ_2 have an end point in common

or (2) Δ_1 and Δ_2 have an interval between them of length $< \frac{1}{2}\epsilon\Delta_1$

We can similarly proceed with the right end point of Δ_2 getting either

(1) an interval Δ_3 abutting on Δ_2

or (2) an interval Δ_3 separated from Δ_2 by an interval δ_2 (say) of length less than $\frac{1}{2}\epsilon\Delta_2$.

By continuing the process we may reach b, or we may not. In the latter case, the chain of intervals formed by continuing the process indefinitely comes to an end at some point ξ in (a, b) We can number the Δ intervals from the beginning as Δ_1 Δ_2 ...

(3 2) In case ξ is a point of CG, the process can be begun again from ξ giving us another chain of intervals If ξ is not a point of CG, we can construct an interval of length $\leq \frac{1}{2^2} \epsilon \Delta$, (where ϵ is any finite number) with ξ as left end point and a point ξ' of CG as right end point The process can now begin with ξ'

Thus between two successive chains there may be a δ interval $(\delta_{\omega+1})$ followed by an interval $\Delta_{\omega+1}$

Similarly we may have in the next chain $\delta_{\omega 2+1}$ followed by $\Delta_{\omega 2+1}$ and so on These chains themselves may have a limiting point, say ξ''

(3 3) We shall call a Δ interval a used-up Δ interval in case it has been employed an infinite number of times to obtain δ intervals in such a way that the sum of these δ intervals is equal to $\epsilon \Delta$ We observe that the selection of δ intervals may be so arranged that all the Δ intervals of the first chain are not used up in reaching ξ'' Further the selection can be such that the Δ intervals of all the other chains except the first are not employed at all, or employed only once

If ξ'' is not a point of CG we may associate with ξ'' a δ-interval employing any one of the infinite un-used Δ intervals preceding ξ'' Beginning from the right end of this interval which we may denote as δ_{ω^2+1}, we can proceed further till we arrive at ξ''' a limiting point of chains We observe that every such limiting point is preceded by an infinite number of un-used Δ intervals, for it is the limiting point of chains

(3 4) Continuing in this manner we may arrive at a point ξ^{ω} The intervals from a to ξ^{ω} when numbered will use up all the numbers of the second class up to ω^{ω} In arriving at ξ^{ω} we may so arrange that only $\left(\frac{1}{2} \Delta_k \epsilon + \cdot + \frac{1}{2^n} \Delta_k \epsilon\right)$ of some or all of the Δ_k intervals is used We may now suppose $\frac{1}{2^{n+1}} \Delta_k \epsilon$ to be divided into intervals just as (a, ξ^{ω}) is divided by proportionate reduction. Taking $k=1$,

we have ω^w intervals in an interval of length $\frac{1}{2^{n+1}}\Delta_1\epsilon$ These interval may now be used as δ intervals after ξ^w, the Δ interval following ξ^w ($\Delta\omega_{w+1}$ say) not being used at all Continuing the process we can reach up to any number λ of the second class, using un-used Δ intervals to obtain δ intervals by suitable methods, as every limiting point ξ^{λ_1} is preceded by un-used Δ intervals (As already explained, we call a Δ interval *un-used* as long as the sum of δ intervals constructed on the basis of Δ is less than $\Delta\epsilon$) The possibility of continuing the chain after every limiting point ξ^λ has thus been demonstrated

(3 5) The chain of intervals must reach b before every number λ of the second class is used up, for if it were not so, the number of intervals required to be constructed in (a, b) would be unenumerable, which is impossible

Therefore, a chain of Δ and δ intervals reaching from a to b exists in (a, b)

Now, if the sum of the Δ intervals

$$m_e\Delta=\sum\Delta=k,$$

then, the sum of the δ intervals

$$m_e\delta\leq\kappa\epsilon$$

But κ cannot be greater than $b-a$

Therefore,

$$m_e\delta\leq\kappa\epsilon$$

can be made less than an arbitrarily assigned number η by choosing ϵ small enough And then

$$m_e\Delta\leq b-a-\eta$$

Thus the lemma is proved

(3 6) *Alternative proof* The method of constructing a chain from a to b indicated above has not been made unique by specifying some rule for the selection of intervals All that has been demonstrated is the possibility of the construction and hence the existence of the chain. Various methods of construction may be devised As an illustration we give below a method of constructing the chain which provides an alternative proof of the lemma

Construction (IA). Let a be a point of CG Then with a as left end point an interval is given Call this interval Δ_1 If the right end point of Δ_1 is a point of CG, we have another interval Δ_2 If the right end point of Δ_1 is not a point of CG, we can find a point of CG such that the distance between it and the right end of Δ_1 is less than $\frac{1}{2^2}\epsilon\Delta_1$. With this point of CG we have now an interval Δ_2

We have thus between the two intervals Δ_1 and Δ_2 an interval, say δ_1, such that $\delta_1 < \frac{1}{2^2}\epsilon\Delta_1$ Proceeding as above we find another interval Δ_3 such that (*a*) either Δ_2 and Δ_3 abut on each other or (*b*) between Δ_2 and Δ_3 there is an interval, say δ_2, such that $\delta_2 < \frac{1}{2^3}\epsilon\Delta_1$ Repeating this construction indefinitely we get a chain of intervals from a to x In case the point x coincides with b, we have obtained the required chain If it does not, then we can begin again with x in case it is a point of CG.

Even if x were not a point of CG, we can obtain a point x' of CG, such that $(x'-x)<\frac{1}{2}\epsilon\Delta_1$ and begin with x' as before

We thus demonstrate the possibility of (1) constructing a chain of intervals from a to x composed of Δ and δ intervals, such that the sum of the δ intervals of the chain $<\frac{1}{2}\epsilon\Delta_1$, and (2) extending the construction beyond x by the use of a δ interval of length $<\frac{1}{2}\epsilon\Delta_1$

Let the intervals of the chain a to x be numbered from the beginning By doing so we shall obtain all the numbers of the first class 1, 2, 3, and the point x will correspond to the number ω, the first number of the second class

Construction (IB) Let now the points of the interval (a, x) be made to correspond to the points of the interval $\left(\frac{1}{2^r}\Delta_1\epsilon, \frac{1}{2^{r+1}}\Delta_1\epsilon\right)$ so that corresponding to the intervals in (a, x) there is a set of sub-intervals in $\left(\frac{1}{2^r}\Delta_1\epsilon, \frac{1}{2^{r+1}}\Delta_1\epsilon\right)$ Let this construction be performed for all values of r Imagine intervals of type $\left(\frac{1}{2^r}\Delta_1\epsilon, \frac{1}{2^{r+1}}\Delta_1\epsilon\right)$, obtained by letting $r=0, 1, 2, \ldots$, arranged in order of magnitude in $(0, \Delta_1\epsilon)$, and let their sub-intervals obtained as above be numbered.

We shall thereby obtain all numbers of the first class and all the numbers of the second class up to ω^2, the point $\Delta_1\epsilon$ corresponding to ω^2

Construction (IIA) The construction (A) may now be performed again, beginning from a with the difference that the δ intervals will now be less than the corresponding ω^2, sub-intervals beginning from the point $\frac{1}{2}\Delta_1\epsilon$ in $(\frac{1}{2}\Delta_1\epsilon, \Delta_1\epsilon)$

We observe that a new series of chains of intervals reaching from a to some point x' can be obtained by using the intervals obtained as under (IB) The intervals when numbered from a onwards will correspond to the numbers of the first class and all the numbers of the second class up to ω^2 which corresponds to the point x' The chain may be further extended from x' to another point ξ by using the intervals in $(0, \frac{1}{2}\Delta_1\epsilon)$ in (IB), but we stop at x'

We further observe that the sum of all the δ intervals in the chain a to x' is less than $\frac{1}{2}\Delta_1\epsilon$.

Construction (IIB) The interval (a, x') being placed in correspondence with each of the intervals $\left(\frac{1}{2^r}\Delta_1\epsilon, \frac{1}{2^{r+1}}\Delta_1\epsilon\right)$ as in (IB) and numbered from the point $\frac{1}{2}\Delta_1\epsilon$ onwards gives us intervals which correspond to numbers of the second class up to ω^3.

These intervals may again be used to construct a chain from a to some point x'', whose intervals when numbered use up all the numbers of the second class up to ω^3

The above process may be supposed to be repeated so many times that a chain from a to b is obtained

We observe that the process of construction employed enables us to ascend higher and higher in the series of numbers of the second class * The process of construction, therefore, must give us a chain reaching from a to b, otherwise the intervals required to be constructed would be unenumerable, which cannot be the case.

It has been thus shown that a chain composed of Δ and δ intervals can be constructed reaching from a to b and such that the sum of

* For, if the chain from a to x corresponds to the number λ of the second class, the construction (B + A) enables us to reach the number $\lambda\,\omega$ which is ordinarily greater than λ The process of construction can go on indefinitely because the chain a to x is greater than Δ_1 and hence always >0 in length, whatever be the number λ to which x corresponds This holds also for the intervals obtained under construction B

the δ intervals of the chain is less than $\frac{1}{2}\Delta_1\epsilon$ Therefore, the sum of the Δ intervals is greater than $b-a-\frac{1}{2}\Delta_1\epsilon$ As ϵ is arbitrary, the lemma is true

Applications to absolutely continuous functions

4 *Theorem I If one of the unilateral derivates of an absolutely continuous function $f(x)$ defined in an interval (a, b) be known to be zero at each point of a set CG which is complimentary to set G of the first category in (a, b), then $f(x)$ is a constant in (a, b)*

Let us consider an interval (a, ξ), $\xi<b$ As it is known that at each point x of a set CG_ξ in (a, ξ), one of the derivates is zero, it is possible to construct with each point of CG_ξ an interval $(x, x+h)$ such that

$$(4\,1) \qquad \left| \frac{f(x+h)-f(x)}{h} \right| \leq \epsilon.$$

Let h have the greatest possible value so that the inequality (4 1) is satisfied Thus to each point of CG_ξ there corresponds an uniquely determined Δ interval such that the difference between the functional values at the end points, F_Δ, is less than $\Delta\epsilon$.

Now by the Lemma, there exists a chain from a to ξ composed of Δ and δ intervals such that $m_e\Delta \geq (\xi-a)-\eta$ and $m_e\delta<\eta$, where η is an arbitrarily small positive number, which tends to zero with ϵ,

We have now,

$$f(\xi)-f(a) \leq \Sigma F_\Delta + \Sigma F_\delta,$$

where F_δ denotes the difference between the values of $f(x)$ at the ends of a δ interval

Therefore

$$(4\,2) \qquad f(\xi)-f(a) \leq \epsilon\Sigma\Delta + \Sigma F_\delta$$

$$\leq \epsilon\,(\xi-a-\eta) + \Sigma F_\delta$$

Now ΣF_δ is the variation of $f(x)$ over a set of intervals δ whose measure

$$m_e\delta<\eta$$

Since η tends to zero as ϵ tends to zero, and $f(x)$ is absolutely continuous,

$\sum F_\delta$ must tend to zero as $\epsilon \longrightarrow 0$

Thus $f(\xi)-f(a)=0$, because ϵ can be taken as small as we please

$$\therefore \quad f(\xi)=f(a)=f(b)$$

(4 3) *Corollary If $f(x)$ has a zero unilateral derivate at all points of a set complimentary to a set of the first category, either $f(x)$ is a constant or $f(x)$ is non absolutely continuous*

(4.4) *Corollary If $f(x)$ be absolutely continuous in (a, b) and if at a set CG complimentary to a set G of the first category in (a, b) $D^+f(x)\geq 0$, then $f(x)-f(a)\geq 0$ Similar results hold for the other derivates*

This follows from the proof of the above theorem

5. *Theorem II If two absolutely continuous functions $f_1(x)$ and $f_2(x)$ have the same upper (lower) derivate on the right (left) at all points of a set CG complimentary to a set G of the first category, in (a, b), then $f_1(x)=f_2(x)+k$ in (a, b), where k is a constant*

We have, for a sequence of $h \longrightarrow 0$,

$$(5\ 1) \qquad \frac{f_1(x+h)-f_1(x)}{h} > D^+f_1(x)-\epsilon,$$

and for all sufficiently small values of h

$$(5\ 2) \qquad \frac{f_2(x+h)-f_2(x)}{h} < D^+f_2(x)+\epsilon$$

Subtracting, as $D^+f_1(x)=D^+f_2(x)$,

we have

$$\frac{\{f_1(x+h)-f_2(x+h)\}-\{f_1(x)-f_2(x)\}}{h} > -2\epsilon$$

Putting $f(x)=f_1(x)-f_2(x)$, we have

$$\frac{f(x+h)-f(x)}{h} > -2\epsilon,$$

for a certain sequence of values of h tending to zero,

or $\qquad D^+f(x)\geq 0$ as ϵ is arbitrary

By interchanging $f_1(x)$ and $f_2(x)$ in the above argument, we see that

$$D^+\{-f(x)\}\geq 0$$

Hence $D^+f(x)=0$ at all points where $D^+f_1(x)=D^+f_2(x)$

Now, the function $f(x)=f_1(x)-f_2(x)$ is absolutely continuous and as $D^+f(x)=0$ at the points of CG, hence by Theorem I, $f(x)$ is a constant$=k$ say,

$$f_1(x)=f_2(x)+k$$

(5 3) *Corollary If two continuous functions $f_1(x)$ and $f_2(x)$ have the same upper (lower) derivate on the right (left) at each point of a set CG complementary to a set G of the first category, then either $f_1(x)-f_2(x)$ is equal to constant or $f_1(x)-f_2(x)$ is a non-absolutely continuous function*

6 *Theorem III If $f(x)$ be an absolutely continuous function in (a, b), then the set of points where*

$$\left.\begin{array}{r} D^+f(x)>k \\ \text{and } D_+f(x)<k \end{array}\right\} (6\ 1)$$

cannot be a set CG complementary to a set G of the first category in (a, b) Similar result holds for the derivates on the left

If the condition (6 1) holds at all points of CG, then, by the reasoning of Theorem I it will be possible to show that $f(b)-f(a)>k(b-a)$ and also $f(b)-f(a)<k(b-a)$ which is absurd Hence the theorem

The above theorem can be extended to the case of any two derivates For example *it can be shewn that*

$$\left.\begin{array}{r} D^+f(x)>k \\ \text{and } \quad D_-f(x)<k \end{array}\right\}$$

cannot hold at all points of a set CG complementary to a set G of the first category

(6 2) *Corollary If $f(x)$ be an absolutely continuous function in (a, b), then $D^+f(x)$ cannot be greater than one of the other three derivates at all points of a set CG complementary to a set G of the first category.*

Applications to functions of bounded variation

7 Let $f(x)$ be a monotone non-diminishing function in (a, b) Then the total variation of the function in (a, b) is $f(b)-f(a)=V_a^b$ (say) The following theorem is seen to hold for $f(x)$

Theorem IV A monotone non-diminishing function $f(x)$ defined in (a, b) cannot have one of its derivates greater than $\left\{\frac{f(b)-f(a)}{(b-a)}\right\}$ at each point of a set CG which is complementary to a set G of the first category in (a, b)

With each point x of CG let us associate the greatest Δ interval $(x, x+\Delta_x)$ such that

$$\frac{f(x+\Delta_x)-f(x)}{\Delta_x} \geq M$$

$$\text{or} \qquad f(x+h)-f(x) \geq \Delta_x M,$$

where M is the lower boundary of one of the eight derivates at points of CG

(7 1) Then by Lemma II, there exists a chain of Δ and δ intervals such that $m_e\Delta > b-a-\eta$ and $m_e\delta < \eta$ where η is arbitrarily small

Therefore,

$$f(b)-f(a)=\Sigma\{f(x+h)-f(x)\}$$

$$=\Sigma F_\Delta+\Sigma f_\delta$$

where ΣF_Δ denotes $\Sigma\{f(x+\Delta_x)-f(x)\}$, x being the left end point of the interval Δ_x The symbol Σf_δ has a similar meaning Consequently

$$f(b)-f(a) \geq \Sigma M\Delta_x+V_\delta$$

where Δ_x denotes the length of the Δ interval satisfying (7 1) and V_δ denotes Σf_δ, the variation of $f(x)$ over the δ intervals. We note that $V_\delta \geq 0$ because $f(x)$ is monotone non-diminishing

Therefore,

$$f(b)-f(a) \geq M\Sigma\Delta_x$$

$$\geq M\{(b-a)-\eta\}$$

$$\geq M(b-a) \tag{7.2}$$

as η is arbitrary

Now, if $$M(b-a) > f(b)-f(a)$$

$$\text{or } M > \frac{f(b)-f(a)}{(b-a)},$$

(7 2) will not be true, and there will be a contradiction. It follows, therefore, that one of the derivates of $f(x)$ cannot be greater than $\dfrac{f(b)-f(a)}{(b-a)}$ at all points of a set CG complementary to a set of the first category G in (a, b)

(7 3) *Corollary If $f(x)$ be monotone decreasing in (a, b), then at each point of a set CG complementary to a set of the first category, one of the derivates cannot be less than $\dfrac{f(b)-f(a)}{(b-a)}$ In fact the derivates at each point x of CG are such that*

$$\frac{f(b)-f(a)}{b-a} \leq Df(x) \leq 0$$

(7 4) *Corollary A monotone non-diminishing (non-increasing) function $f(x)$ cannot have one of its derivates $+\infty$ $(-\infty)$ at each point of a set CG complementary to a set G of the first category*

This follows directly from Theorem IV

(7 5) Examples of monotone functions having infinite derivates at everywhere dense sets have been constructed by Brodén,* and others. According to a theorem of W H Young,† the set of points where one of the derivates is infinite is an inner limiting set, so that the set of points where the monotone function of Brodén possesses infinite derivates is of the second category But by corollary (7.4) this set cannot be complementary to a set of the first category, so that there exists in the case of Brodén's function, a set of the second category at which the function has finite derivates By a well known theorem about the derivates of functions of bounded variation this set of the second category has a measure equal to that of the whole interval.

* Brodén, *Crelle's Journal,* Vol CXVIII (1897), also *Acta Univ. Lund.*, Vol XXXIII (1897)

† Young, *l c*

Thus Brodén's functions may be used to divide a given interval (a, b) into two sets of the second category, one of which has zero measure

(7 6) *Corollary A monotone increasing (decreasing) function $f(x)$ cannot have a zero derivate at all points of a set CG which is complementary to a set G of the first category*

The function $\phi(x)=\frac{1}{f(x)}$ is also single valued and monotone, and the derivates of $\phi(x)$ are infinite at all points where $f(x)$ has zero derivates. But by corollary (7 4) $\phi(x)$ cannot have infinite derivates at all points of CG Hence the result

Examples of monotone increasing or decreasing function with zero derivates at everywhere dense sets may be constructed Such functions may also be used to divide a given interval into two sets of the second category.

8 *Theorem V If $f(x)$ be a function of bounded variation in (a,b), whose positive variation is P_a^b in (a, b), then $f(x)$ cannot have at each point of a set CG complementary to a set G of the first category, a derivate greater than $\frac{P_a^b}{b-a}$*

Let $f(x)$ be a function of bounded variation defined in the interval (a, b) Let V_a^b be its total variation in (a, b) and P_a^b and N_a^b its total positive and negative variations

If possible let one of the derivates of $f(x)$ be $>M$ at each point of the set CG then to each point x of CG we associate the greatest Δ interval $(x, x+\Delta_x)$ such that

$$\text{(8 1)} \qquad \frac{f(x+\Delta_x)-f(x)}{\Delta_x}\geq M.$$

Then there is a chain of Δ and δ intervals from a to b and such that

$$f(b)-f(a)=\Sigma F_\Delta+\Sigma f_\delta$$

where F_Δ and f_δ have the same meanings as in the previous theorem

It follows that

$$f(b)-f(a)\geq M\Sigma\Delta_x-N_a^b$$

$$\geq M(b-a-\eta)-N_a^b$$

(8 2) or $$\{f(b)-f(a)\}+N_a^b \geq M(b-a)$$

as η is arbitrary Now, (8 2) will not hold if

$$M > \frac{f(b)-f(a)+N_a^b}{b-a},$$

$$> \frac{P_a^b}{b-a}$$

for $$f(b)-f(a)=P_a^b-N_a^b$$

Therefore, $f(x)$ cannot have a derivate which is greater than $\frac{P_a^b}{(b-a)}$ at a set CG complementary to a set G of the first category in (a, b)

9 *Theorem VI If $f(x)$ be of bounded variation in (a, b) and if N_a^b be its total negative variation in (a, b), then $f(x)$ cannot have at each point of a set CG complementary to a set G of the first category in (a, b), a derivate which is less than* $\frac{-N_a^b}{(b-a)}$

The proof of this result is on the same lines as in the previous theorem, for in this case the inequality (8 1) and (8 2) of the previous theorem can be replaced respectively by

$$\frac{f(x+\Delta_x)-f(x)}{\Delta_x} \leq M$$

and $$f(b)-f(a)-P_a^b \leq M(b-a),$$

so that there is a contradiction if

$$M < \frac{f(b)-f(a)-P_a^b}{(b-a)}$$

$$< -\frac{N_a^b}{(b-a)}$$

(9 1) *Corollary. The set of points where a function of bounded variation in (a, b) possesses an infinite derivate with fixed sign ($+\infty$ or $-\infty$) cannot be a set of the second category of the type which is the complement of a set of the first category*

For, if one of the derivates is $+\infty$, Theorem V applies because $\frac{f(b)-f(a)+N_a^b}{(b-a)}$ is finite, so that one of the derivates cannot be $+\infty$ at all points of CG In the same way by Theorem VI, one of the derivates cannot be $-\infty$ at the set CG

9 2 It is known that the set of points where a function of bounded variation has an infinite derivate is of zero measure It is also known that the set is an inner limiting set and so is of the second category when it is everywhere dense The above corollary shows that this set cannot be the complement of a set of the first category

It is possible that a function of bounded variation may have (1) one of its derivates equal to $+\infty$ at a set S_1 of the second category, (2) one of its derivates equal to $-\infty$ at a set S_2 of the second category, (3) one of its derivates finite at a set s_3 of the second category

In the above S_1 and S_2 will in general have common points In case they are the same, the set S_3 must be of the second category In case S_1 and S_2 are not identical it seems that S_3 may be of the first category But neither S_1 nor S_2 is the complement of a set of the first category It will now be shown that S_3 is also of the second category

10 *Theorem VII If $f(x)$ is of bounded variation in (a, b), then one of the right (left) derivates of $f(x)$ cannot be in absolute value greater than $V_a^b/(b-a)$ at each point of a set CG complementary to a set G of the first category in (a, b)*

If one of the derivates on the right at each point of CG is in absolute value greater than M, then, to each point x of CG we can associate a Δ interval $(x, x+\Delta_x)$ such that

$$(10\ 1) \qquad \left| \frac{f(x+\Delta_x)-f(x)}{\Delta_x} \right| \geq M$$

There is a chain of Δ and δ intervals from a to b, such that $\Sigma\Delta>(a-b-\eta)$ and $\Sigma\delta<\eta$. Therefore,

$$V_a^b \geq M\Sigma\Delta+\Sigma f_\delta,$$

where V_a^b is the total variation of $f(x)$ in (a, b) and Σf_δ is the sum of

the variations of $f(x)$ over the δ intervals. We observe that $\Sigma\Delta \geq (a-b-\eta)$ and Σf_{δ} is positive Therefore

$$V_a^b - \Sigma f_\delta > M(b-a-\eta)$$

$$(10\ 2) \quad \text{or} \qquad V_a^b > M(b-a-\eta)$$

and as η is arbitrary,

$$(10\ 3) \qquad \frac{V_a^b}{b-a} \geq M$$

Now, if $$M > \frac{V_a^b}{(b-a)},$$

there will be a contradiction Hence one of the right hand derivates cannot be greater than $\frac{V_a^b}{(b-a)}$ at each point of CG It can be similarly proved that one of the left hand derivates cannot be greater than $\frac{V_a^b}{(b-a)}$ at each point of CG

(10 4) *Corollary The set of points where one of the right (left) derivates of a function $f(x)$ of bounded variation in (a, b) is infinite in value cannot be a set CG which is the complement of a set G of the first category.*

This follows directly from the above theorem

(10.5) Thus the set S_1 where one of the right derivates is infinite in value is not of the type CG Also the set S_2 where one of the left derivates is infinite in value is not of the type CG It will be shown that the set (S_1+S_2) which contains all the points of S_1 and also of S_2, common points being taken only once, is not of the type CG.

By a well known theorem of W H Young, the set of points where there is distinction of right ~~and left~~ as regards the values of the derivates form a set of the first category Therefore, the points which are not common to S_1 and S_2 form a set of the first category, say g_1 Hence

$$S_1+S_2=S_1\ S_2+g_1$$

But as S_1 and S_2 separately are not of the type CG, so $S_1 . S_2$ is also not of the type CG, because $S_1\ S_2$ is contained both in S_1 and S_2 consequently we have the following result ;

(10 6) *Corollary The set of points where a function $f(x)$ of bounded variation in (a, b) has finite derivates is of the second category*

11 *Theorem VIII If $f(x)$ be a function of bounded variation in (a, b), and of total variation V_a^b then the set of points where*

$$\left.\begin{array}{ll} & D^+f(x) > k_1 \\ \textit{and} & D_-f(x) < k_2 \\ \textit{where} & k_1 - k_2 > \dfrac{2V_a^b}{(b-a)} \end{array}\right\} \qquad (11\ 1)$$

cannot be a set CG complementary to a set G of the first category

For, if the condition (11 1) be satisfied at each point of CG, then with each point x of CG we can have a Δ interval $(x, x+\Delta_x)$ such that

$$(11\ 2) \qquad \frac{f(x+\Delta_x)-f(x)}{\Delta_x} \geq k_1$$

Hence as in Theorem V

$$(11\ 3) \qquad f(b)-f(a) \geq k_1 \Sigma\Delta - \Sigma f_\delta$$

where Σf_δ is the variation of $f(x)$ over the δ intervals Again, with each point x of CG we can have a Δ' interval $(x, x+\Delta'_x)$, such that

$$\frac{f(x+\Delta'_x)-f(x)}{\Delta'_x} \leq k_2,$$

so that we have

$$f(b)-f(a) \leq k_2 \Sigma\Delta' + \Sigma f_{\delta'} \qquad (3)$$

where $\Sigma f_{\delta'}$ is the variation of $f(x)$ over the δ' intervals. Subtracting (3) from (2) we get

$$0 \geq k_1 \Sigma\Delta - k_2 \Sigma\Delta' - \Sigma f_\delta - \Sigma f_{\delta'}$$

But both $\Sigma\Delta$ and $\Sigma\Delta'$ can be made as near $b-a$ as we please, therefore

$$0 \geq (k_1-k_2)(b-a) - \{\Sigma f_\delta + \Sigma f_{\delta'}\}$$

$$i\ e, \ 0 \geq (k_1-k_2)(b-a) - 2V_a^b$$

or

$$\frac{2V_a^b}{(b-a)} > k_1 - k_2$$

which is contrary to (11 1) The theorem, therefore, holds.

12 *Theorem IX* *If $f(x)$ be a function of bounded variation in (a, b), the set of points where $D^+f(x) > k_1$ and any one of the other three derivates is less than k_2, where $k_1 - k_2 > \dfrac{2\{V_a^b + f(b) - f(a)\}}{(b-a)}$ cannot be a set CG complementary to a set G of the first category in (a,b).*

For, if possible, suppose

$$\left.\begin{aligned} & D^+f(x) > k_1 \\ & D_-f(x) < k_2 \\ & k_1 - k_2 > \frac{2\{V_a^b + f(b) - f(a)\}}{(b-a)} \end{aligned}\right\} \tag{12 1}$$

at each point of CG Then the reasoning of the above theorem can be easily applied. In one case we shall have

$$f(b) - f(a) \geq k_1 \Sigma\Delta - \Sigma f_\delta$$

as before, and in the other case

$$f(a) - f(b) \leq k_2 \Sigma\Delta' + \Sigma f_{\delta'}$$

whence

$$2\{f(b) - f(a)\} \geq k_1 \Sigma\Delta - k_2 \Sigma\Delta' - \Sigma f \; - \Sigma f_{\delta'}$$

or $$2\{f(b) - f(a)\} \geq (k_1 - k_2)(b-a) - 2V_a^b$$

(12 2) or $$2\{f(b) - f(a)\} + 2V_a^b > (k_1 - k_2)(b-a).$$

Thus (12 1) and (12.2) are contradictory and, therefore, the theorem holds.

4

An Integral representing Self-Reciprocal Functions

BY

Brij Mohan Mehrotra

1 The object of this note is to obtain a new formula for self-reciprocal functions The interest lies mainly in the result and not in a rigorous proof thereof Hence only the formal procedure is given here.

Following Hardy and Titchmarsh I will say that a function is R_ν if it is self-reciprocal for J_ν transforms, and it is—R_ν if it is skew reciprocal for J_ν transforms Also, for $R_{\frac{1}{2}}$ and $R_{-\frac{1}{2}}$ I will write R_s and R_c respectively.

I will make use of the following result of Hardy and Titchmarsh* —

A necessary and sufficient condition that a function $f(x)$ should be R_ν is that it should be of the form

$$f(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s}\,\Gamma\left(\frac{1}{4}+\frac{1}{2}\nu+\frac{1}{2}s\right)\psi(s)x^{-s}ds, \tag{1 1}$$

where $o<c<1$ and

$$\psi(s)=\psi(1-s). \tag{1 2}$$

Using a form of Mellin's Inversion Formula,† we get, from (1·1)

$$\int_0^\infty x^{s-1}f(x)dx=2^{\frac{1}{2}s}\,\Gamma\left(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s\right)\psi(s),$$

where $\psi(s)$ satisfies (1 2).

* G. H. Hardy and E C Titchmarsh · Self-Reciprocal Functions, Quarterly Journal of Math , Oxford Series, I (1930), 196 231, § 3

† See G. H Hardy Further Notes on Mellin's Inversion Formula, Messenger of Math , 50 (1921), 165-171

2 In a recent paper I have shown that the function

$$g(x)=\int_0^\infty \frac{F(w)}{\sqrt{w}} f(xw)\, dw, \tag{2 1}$$

where $f(x)$ is R_ν and $F(w)$ satisfies the equation

$$F(w)=F\left(\frac{1}{w}\right),$$

is R_ν *

Now, let

$$F(w)=\int_0^1 w^{u-\frac{1}{2}}\, G(u)\, du,$$

where $G(u)=G(1-u)$

Then
$$F\left(\frac{1}{w}\right)=\int_0^1 w^{\frac{1}{2}-u}\, G(u)\,du$$

$$=\int_0^1 w^{s-\frac{1}{2}}\, G(1-s)\, ds$$

$$=\int_0^1 w^{s-\frac{1}{2}} G(s)\,ds$$

$$=F(w),$$

so that $F(w)$ satisfies (2 2).

Thus
$$g(x)=\int_0^\infty \frac{F(w)}{\sqrt{w}} f(xw)\,dw$$

$$=\int_0^\infty \frac{f(xw)}{\sqrt{w}}\, dw \int_0^1 w^{u-\frac{1}{2}}\, G(u)\,du$$

$$=\int_0^1 G(u)\,du \int_0^\infty w^{u-1} f(xw)\,dw,$$

provided that the inversion of the order of integration is justified.

* B M Mehrotra · On some Self reciprocal Functions—Bull., Calcutta Math. Soc., XXV, 1933 (167 172), 170 See also B M Mehrotra Some Theorems on Self-reciprocal Functions, Proc, London Math Soc, Series 2, 34 (1932), 231-[illegible], § 5 (iii), and W N Bailey On the Solution of Some Definite Integral Equations, Journal, London Math Soc., VI (1931), 242 247 Particular cases of (2 1) were given by Hardy and Titchmarsh, ibid, § 3

Hence
$$g(x)=\int_0^1 x^{-u}G(u)\,du\int_0^\infty y^{u-1}f(y)\,dy$$

Using (1 3) we have

$$g(x)=\int_0^1 x^{-u}G(u)\quad 2^{\frac{1}{2}u}\Gamma\left(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}u\right)\psi(u)\,du,$$

where
$$\psi\ (u)=\psi(1-u)$$

Hence
$$g(x)=\int_0^1 2^{\frac{1}{2}s}\Gamma\left(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s\right)\chi(s)x^{-s}\,ds, \tag{2 3}$$

where
$$\chi(s)=\chi(1-s). \tag{2 4}$$

This formula is similar to (1 1) except for a constant multiple and the limits of integration

3 Let us test whether the function given by the integral (2 3) is self-reciprocal or not For simplicity we take the particular case of cosine-transform, that is, the case when $\nu=-\frac{1}{2}$

$$g(x)=\int_0^1 2^{\frac{1}{2}s}\ \Gamma\left(\tfrac{1}{2}s\right)\chi(s)x^{-s}\,ds.$$

The cosine-transform of $g(x)$

$$=\sqrt{\frac{2}{\pi}}\int_0^\infty g\ (y)\cos xy\ dy$$

$$=\sqrt{\frac{2}{\pi}}\int_0^\infty \cos xy\ dy\int_0^1 2^{\frac{1}{2}s}\Gamma\left(\tfrac{1}{2}s\right)\chi(s)y^{-s}\,ds$$

$$=\sqrt{\frac{2}{\pi}}\int_0^1 2^{\frac{1}{2}s}\Gamma\left(\tfrac{1}{2}s\right)\chi\ (s)\ ds\int_0^\infty y^{-s}\cos xy\ dy,$$

provided again that the inversion of the order of integration is justified

Hence the cosine-transform of $g\ (x)$

$$= \sqrt{\frac{2}{\pi}} \int_0^1 2^{\frac{1}{2}s} \Gamma\left(\tfrac{1}{2}s\right)\chi(s)\ \frac{\Gamma(1-s)}{x^{1-s}}\ \cos\frac{(1-s)\pi}{2}\, ds$$

$$= \int_0^1 2^{\frac{1}{2}+\frac{1}{2}s} \Gamma\left(\tfrac{1}{2}s\right) \chi(s) \frac{\sqrt{\pi}\,\Gamma(1-s)}{\Gamma\left(\frac{1}{2}s\right)\Gamma\left(1-\frac{1}{2}s\right)}\, x^{s-1} ds$$

$$= \int_0^1 2^{\frac{1}{2}-\frac{1}{2}s} \Gamma\left(\tfrac{1}{2}-\tfrac{1}{2}s\right)\chi(s) x^{s-1} ds.$$

Putting $s=1-u$, this becomes

$$\int_0^1 2^{\frac{1}{2}u} \Gamma\left(\tfrac{1}{2}u\right) \chi(u) x^{-u} du$$

$$= g(x),$$

which shows that $g(x)$ is R_c

4 If $f(x)$ is $-R_\nu$, the function $g(x)$ given by (2 3) is $-R_\nu$.

But if $f(x)$ is $-R_\nu$ and $\chi(s)$ satisfies the equation

$$\chi(s)=\chi(1-s),$$

instead of (2 4), $g(x)$ is again R_ν

Bull., Cal. Math Soc, Vol. XXVI, No. 1 (1934)

5

Geometrisation of Physics

BY

S C Bagchi.

§ 1

About the middle of the nineteenth century Sophus Lie remarked that the whole of physical nature may be regarded as a system of infinitesimal transformations, the law of nature being the invariant of the transformations. Riemann and Clifford threw out the suggestion that the entity, if there be any, called force in dynamics may be the binding operator in an otherwise amorphous space The Riemann Christoffel tensor is a kinematic quantity appearing as a kinetic quantity in physical equations The structure of modern physics shows the direction in which the great synthesis suggested by Lie lies Theoretical physics is concerned with certain transformations and invariants Geometry gives a local habitation and a name to these. Juvet and Leroy remark, following Duhem, "today every affirmation on the geometry of the universe has its counterpart in electro magnetism." * Relativity, Wave-mechanics and Quantum theory are the physical expressions of theorems of transformation in generalised spaces The elements of a variable phenomenon, constitute a system of co-ordinates, a particular state of the phenomenon, characterised by an aggregate of its co-ordinates, is called a point, the aggregate of points forms a variety, in other words the whole history of a phenomenon is imaged as a variety—a manifold of punctual aggregate The aggregate of variable phenomena may be studied as an aggregate of geometrical varieties

* French translation of Weyl's Raum Zeit, Materie, Introduction

The geometrisation of physics started in a desire to formulate a unitary physical theory. In classical mechanics Euclidean spaces with a finite number of dimensions were utilised for representing certain solutions of Jacobi-Hamilton equations The extensional calculus of Grassmann and Hamilton leading on to the tensor analysis of the present century has consolidated this process of geometrisation In the generalised geometry of physical phenomena the idea of groups of transformations as well as that of matrix in generalised space plays a fundamental part So our history will begin with the groups of Lie and spaces of Cartan.

The fundamental structural relations in the theory of finite and continuous groups are

$$(1) \quad c_{ij}^{s} + c_{ji}^{s} = 0$$

$$(2) \quad c_{ij}^{a} c_{ka}^{s} + c_{jk}^{a} c_{ia}^{s} + c_{ki}^{a} c_{ja}^{s} = 0$$

By contraction (2) becomes

$$(3) \quad c_{ij}^{a} c_{sa}^{s} = 0, \ a, s \text{ being umbral}$$

This form of (2) shows that if the constants c_{ij}^{s}, for which $i, j, s = 1, 2, \ldots r$ satisfy (1) and (2) one may always find r infinitesimal transformations x_s, such that

$$(4) \quad (X_i X_j) = X_i X_j - X_j X_i = c_{ij}^{s} X_s$$

This is the famous third theorem of Lie. Now Buhl has shown that starting from a generalised Stokes' theorem we arrive at the unitary field theory of Einstein through this third theorem of Lie.*

Take the identical relations

$$(5) \quad \int_{C} X dY \equiv \iint_{S} dX dY, \quad \iint_{S} X dY dZ \equiv \iiint_{V} dX dY dZ$$

and all the analogous relations with any number of variables $x, y, \ldots$ The two identities put down give the gravitational forms of Einstein for the ordinary space-time By successive changes of variables and linear combinations the identities (5) lead to Maxwell's electromagnetic equations So the basis of electromagnetic theory is a kind of geometry. This result is obtained most readily by utilising Pfaffian forms and

* See Buhl-Gravifique, &c, p 4

the method of external derivation of Cartan The identities (5) are then replaced by

$$(6)\quad \int_l P_i\, dx^i \equiv \iint_s [dP_i\, dx^i]$$

$$\iint_s M_{ij}\, dx^i dx^j \equiv \iiint_v [dM_{ij}\, dx^i dx^j]$$

The developments of the brackets give the symbolic determinants

$$(7)\quad \begin{vmatrix} \dfrac{\partial}{\partial x^i} & \dfrac{\partial}{\partial x^j} \\ P_i & P_j \end{vmatrix},$$

$$(8)\quad \begin{vmatrix} \dfrac{\partial}{\partial x^i} & \dfrac{\partial}{\partial x^j} & \dfrac{\partial}{\partial x^k} \\ M_{iw} & M_{ju} & M_{kw} \\ i & j & k \end{vmatrix}$$

with $M_{ij} + M_{ji} = 0$

Starting from (7) and replacing ∂ by the more general symbol D we put

$$(9)\quad \begin{vmatrix} \dfrac{D}{Dx^i} & \dfrac{D}{Dx^j} \\ P_i & P_j \end{vmatrix} = \begin{vmatrix} \dfrac{\partial}{\partial x^i} & \dfrac{\partial}{\partial x^j} \\ P_i & P_j \end{vmatrix} - \begin{vmatrix} \Gamma^{\alpha}_{wi} & \Gamma^{\alpha}_{wj} \\ {}_iP_\alpha & {}_jP_\alpha \end{vmatrix}$$

Writing

(10) $\Lambda^{\alpha}_{ij} = \Gamma^{\alpha}_{ij} - \Gamma^{\alpha}_{ji}$, the last determinant in (9) becomes, sign apart,

(11) $\Lambda^{\alpha}_{ij} P_\alpha$

Whatever may be this expression (11), one may always break it up into expressions like

$$(12)\quad \frac{DP_j}{Dx^i} = \frac{\partial P_j}{\partial x^i} - L^{\alpha}_{ij} P_\alpha.$$

Similarly for contravariant vectors P^i one gets with

$$(13)\quad \begin{vmatrix} \frac{D}{Dx^i} & \frac{D}{Dx^j} \\ P^i & P^j \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial x^i} & \frac{\partial}{\partial x^j} \\ P^i & P^j \end{vmatrix} + \begin{vmatrix} \Gamma^{i}_{\alpha j} & \Gamma^{j}_{\alpha i} \\ P^\alpha & P^\alpha \end{vmatrix}$$

where the last determinant on the right-hand is equal to

$$\left(\Gamma^{b}_{\alpha i} - \Gamma^{b}_{\alpha j} \right) P^\alpha$$

which gives

$$(14)\qquad \frac{DP^j}{Dx^i} = \frac{\partial P^j}{\partial x^i} + \Gamma^{j}_{\alpha i} P^\alpha \text{ and}$$

$$(15)\qquad \frac{D}{Dx^i}(P_j P^j) = P_j \frac{DP^j}{Dx^i} + P^j \frac{DP_j}{Dx^i}$$

Thus the parallel displacements of P_i or P^i are difined by

$$(16)\qquad \frac{DP_j}{Dx^i} dx^i = dP_j - \Gamma^{\sigma}_{ji} P_\sigma dx^i = 0$$

$$(17)\qquad \frac{DP^j}{Dx^i} dx^i = dP^j + \Gamma^{j}_{i\alpha} P^\alpha dx^i = 0$$

The rule of derivatives given by (12) and (14) may be put in the general form

$$(18)\qquad \frac{D}{Dx^i} A^{***}_{\alpha **} = \frac{\partial}{\partial x^i} A^{***}_{***} \left\{ -\Gamma^{\alpha}_{\mu i} A^{***}_{**\mu} + \Gamma^{\mu}_{\alpha i} A^{**\alpha *}_{x**\cdots} \right.$$

$-$ or $+$ to be taken according as the covariant or contravariant index occurs, the index of derivation is put last in the series of the lower indices of Γ with the help of the rule (18) the identity of Bianchi may be written at once in a generalised form *viz*

$$(19)\qquad \left(B^{\alpha}_{\mu\nu\sigma} \right)_\tau + \left(B^{\alpha}_{\mu\sigma\tau} \right)_\nu + \left(B^{\alpha}_{\mu\tau\nu} \right)_\sigma$$

$$+ B^{\alpha}_{\mu\nu\beta} \Lambda^{\beta}_{\sigma\tau} + B^{\alpha}_{\mu\sigma\beta} \Lambda^{\beta}_{\tau\nu} + B^{\alpha}_{\mu\tau\beta} \Lambda^{\beta}_{\nu\sigma} = 0$$

Now in the first form of general relativity the gravitational equation of Einstein considers a space which is curved but is not twisted, *i e*, Λ's are zero in (19), *i e*, the identity of Bianchi plays the fundamental role here In the unitary theory the space is twisted but without curvature The equations in the new theory are

$$(20)\qquad (B^{\alpha}_{\nu\sigma\tau}+B^{\alpha}_{\sigma\nu\tau}+B^{\alpha}_{\tau\nu\sigma})+(\Lambda^{\alpha}_{\nu\sigma\tau}+\Lambda^{\alpha}_{\sigma\tau\nu}+\Lambda^{\alpha}_{\tau\nu\sigma})$$

$$+\Lambda^{\beta}_{\nu\sigma}\Lambda^{\alpha}_{\tau\beta}+\Lambda^{\beta}_{\sigma\tau}\Lambda^{\alpha}_{\nu}+\Lambda^{\beta}_{\tau\nu}\Lambda^{\alpha}_{\sigma\beta}=0$$

$$(21)\qquad \Lambda^{\alpha}_{i;j}+\Lambda^{\alpha}_{j;i}=0$$

The different types of gravitational equations, in other words, the different field equations are narrowly connected with different types of group spaces Cartan has defined the components of curvature and torsion by

$$(22)\qquad \Omega^{i}=[\pi^{i}]'-[\pi^{\alpha}\pi^{i}_{\alpha}]=\Lambda^{i}_{jk}[\pi^{j}\pi^{k}]$$

$$(23)\qquad \Omega^{i}_{k}=[\pi^{i}_{k}]'-[\pi^{\alpha}_{k}\pi^{i}_{\alpha}]=B^{i}_{kml}[\pi^{m}\pi^{l}]$$

If we put $\pi^{i}=-dx^{i}$, $\pi^{i}_{j}=\Gamma^{i}_{j\beta}dx^{\beta}$, noting that the accents in (22) and (23) denote exterior derivatives, we get

$$(24)\qquad \Gamma^{i}_{\alpha\beta}-\Gamma^{i}_{\beta\alpha}=\Lambda^{i}_{\alpha\beta}$$

This is the same as the equation (10), and further we have

$$(25)\qquad B^{i}_{kml}=\frac{\partial}{\partial x^{m}}\Gamma^{i}_{kl}-\frac{\partial}{\partial x^{l}}\Gamma^{i}_{km}+\Gamma^{\beta}_{kl}\Gamma^{i}_{\beta m}-\Gamma^{\beta}_{km}\Gamma^{i}_{\beta l}$$

which is the 4-index symbol of Riemann

The groups in Cartan space show that physics is geometrisable as soon as the physical laws are shown to be expressible by means of differential equations connecting the components of torsion and curvature It may be noticed in this connexion that Cartan space with its curvature and torsion may be put in correspondence with torsion-less space in an infinity of ways.

Let there be an n-uple of orthogonal congruences. Through each point P of the space V_n pass n lines orthogonal two by two, numbered

by the lower indices in the subjoined table

$$\begin{array}{cccc} \lambda_1^1 & \lambda_1^2 & \ldots & \lambda_1^n \\ \lambda_2^1 & \lambda_2^2 & & \lambda_2^n \\ \ldots & \ldots & & \ldots \\ \lambda_n^1 & \lambda_n^2 & & \lambda_n^n \end{array}$$

Each row of this defines the parameters of direction of one and the same line of the n-uple Let the table of moments be given by

$$\begin{array}{ccc} \lambda_1^1 & \lambda_2^1 & \lambda_n^1 \\ & \cdot & \\ \lambda_1^n & \lambda_2^n & \lambda_n^n \end{array}$$

The correspondence between these two tables is given by

(26) $$\lambda_i^h \lambda_h^i = \delta_k^k, \ \lambda_i^h \lambda_k^j = \delta_i^j$$

If one puts

$$g^{ik} = \lambda_i^h \lambda_k^h, \ g^{ik} = \lambda_h^i \lambda_h^k$$

one gets

$$g_k^{\cdot} = g^{ir} g^{ij} = \delta_k^j g^{ij} \lambda_i^h = \lambda_h^j, \ g^{ij} \lambda_h^j = \lambda_i^h$$

This allows us to construct a Riemannian metric for which $ds^2 = g_{ik} dx^i dx^k$. Hence we can unite the law of gravitation in Riemannian space with that in Cartan space. Thus the unitary field theory is nothing but the intrinsic geometry of the generalised space

§ 2

Let us go back to (5) of the preceding section. By virtue of $d(XY) = XdY + YdX$ (1) the equation (5) may be written in various forms Now (1) may be written also as

$$\frac{d}{dX}(XY) - X\frac{d}{dX}Y = Y$$

which shows the existence of symbols q and p such that

(27) $$qp - pq = \frac{ih}{2\pi} \cdot 1$$

where h is Planck's constant This equation (2) is at the basis of new mechanics and the equation (1) puts it in accord with the process of geometrisation we are in search after While the classical mechanics utilised spaces with finite number of dimensions the wave mechanics considers wave functions ψ in a functional space of infinite number of dimensions In all the spaces we are concerned with linearity of connections and operators Following Struik a summary account of Hermitian connections may be given here just to point out the geometrical structure of the quantum theory The variables x^n of an E_n can be made to run through all complex values It is easy to establish connexions of the type

$$v^{\bar{\kappa}'}=A^{\bar{\kappa}'}_{\bar{\lambda}}v^{\bar{\lambda}},\ v^{\kappa'}=A^{\kappa'}_{\kappa}v^{\lambda},\ \text{where } A^{\bar{\kappa}'}_{\bar{\kappa}} \text{ is conj } A^{\lambda'}_{\lambda}$$

Next we introduce quantities in which some indices refer to the v^{λ}—space and some to $v^{\bar{\kappa}}$—space, e g ,

$$g_{\lambda'\bar{\mu}'}=A^{\lambda\bar{\mu}}_{\lambda'\bar{\mu}'}g_{\lambda\bar{\mu}},\ A^{\lambda\bar{\mu}}_{\lambda'\bar{\mu}'}=A^{\lambda}_{\lambda'}A^{\bar{\mu}}_{\bar{\mu}'}$$

such quantities are called Hermitian. In a linear space with Hermitian connexion we can construct Hermitian tensors and densities similar to those defined by real variables In investigations connected with the spinning electron it has been shown by Struik that we obtain a geometry of Hermitian quantities in an E_4 by starting from a Minkowskian R_4 In such an R_4 there exists a hyper-sphere

$$g_{ij}x^i x^j=1,\ i,j=0,1,2,3$$

$$-g_{00}=g_{11}=g_{22}=g_{33}=+1$$

$$g_{ij}=0\ i\neq j$$

of signature $-+++$

The ∞^3 straight lines of this hyper sphere determine the directions of ∞^4 vectors These vectors can be represented by the ∞^4 points of an auxiliary E_4 the so called spin-space With this spin-space we come to spin-operators These operators are invariantive in themselves and the representation of the space is independent of the choice of coordinates. A generalised intrinsic geometry is automatically constructed with the help of the spin operators The equations of

Dirac and Schrodinger in spin-space and wave-space may be interpreted as Schrodinger himself has shown with the help of the spinor Let ψ be a spinor and let $T^{\rho\sigma}_{\alpha\beta}\ .\psi=0$. (1) in every reference system so that for a point transformation ψ remains invariant and for an S-transformation it is transformed as follows

$$\psi'=S^{-1}\psi \qquad \dots \quad (2)$$

from (1) by left multiplication by S^{-1}

$$S^{-1}T^{\rho\sigma\dots}_{\alpha\beta\dots}\ SS^{-1}\psi=T'^{\rho\sigma}_{\alpha\beta}\psi'=0 \qquad \dots \quad (3)$$

By a supplementary infinite point transformation we get

$$\psi'=\psi-\Theta\psi \qquad \dots \quad (4)$$

where Θ is the Hermitian operator given by

$$\Theta=-\frac{1}{2g_{00}}a^l_0\gamma_l\gamma_0 \qquad (5)$$

the quantities γ satisfy an equation of the type

$$\gamma'_0\gamma'^k=\gamma_0\gamma^k-a^l_0\gamma_l\gamma^k+a^k_l\gamma_0\gamma^l+\gamma_0\gamma^k\Theta-\Theta\gamma_0\gamma^k \qquad .. \quad (6)$$

where in a γ-field

$$x'_k=x_k+\delta x_k \qquad \dots \quad (7)$$

and an infinite S-transformation is given by

$$S=1+\Theta,\ S^{-1}=1-\Theta \qquad \dots \quad (8)$$

with the help of (1)...(8) the equation of Dirac may be generalised into

$$\gamma^k\nabla_k\psi=\mu\psi \qquad \dots \quad (9)$$

The above considerations suggest a special type of canal space which allows a direct-geometrical representation by means of generalised Stokes' theorem given by Buhl. We get for canal space

$$\int_{\Sigma} UdP-VdQ=\iint_{\sigma}\left(\frac{\partial V}{\partial P}-\frac{\partial U}{\partial Q}\right)\begin{vmatrix}\alpha & \beta & \gamma\\ P_x & P_y & P_z\\ Q_x & Q_y & Q_z\end{vmatrix}d\sigma \qquad \dots \quad (10)$$

An elementary canal has a quadrilateral section, on its lateral faces P, Q, P+dP, Q+dQ are constant

Calling the bracketed coefficient of the determinant Λ (P, Q), $\Lambda\ d$P, dQ may be writen

$$\Lambda\,(P, Q)\begin{vmatrix} \alpha & \beta & \gamma \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix} d\sigma \tag{11}$$

$$=\frac{\Lambda(P, Q)}{\Theta(XYZ)}\begin{vmatrix} \Phi_x & \Phi_y & \Phi_z \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix}\frac{\Theta(X, Y, Z)\,dS}{\sqrt{\Phi_y^2+\Phi_y^2+\Phi_z^2}}$$

the diaphragm S is defined by $\Phi=0$

Now put

$$\frac{1}{\Theta\sqrt{\Phi_x^2+\Phi_y^2+\Phi_z^2}}\begin{vmatrix} \Phi_x & \Phi_y & \Phi_z \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix}=\begin{cases} \Delta(X, Y, Z) \\ \Delta_1(\Phi, P, Q) \text{ on S} \\ \Delta_1(O, P, Q) \end{cases} \tag{12}$$

Then to find out Λ(P, Q) we have

$$\Lambda(P, Q)\Delta_1(O, P, Q)=1$$

This is how we get

$$\iint_s \Theta dS=\iint_\sigma \begin{vmatrix} \alpha, & \beta, & \gamma \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix}\frac{d\sigma}{\Delta_1(O, P, Q)}$$

A canal surface is propagated like a wave of invariant ΘdS Hence we have a general representation of Schrodinger-Jacobi symbol S To make the geometric aspect of Shrodinger ψ more explicit let us take the equation

$$J(S)=\left(\frac{\partial S}{\partial x}\right)^2+\left(\frac{\partial S}{\partial y}\right)^2+\left(\frac{\partial S}{\partial z}\right)^2-S^2\Omega$$

$$\Sigma(W)=\Delta W+W\Omega \qquad .. \quad (13)$$

Replacing W by ϕ, the Schrodinger equation is

$$(14) \qquad \frac{h^2}{2m}\Delta\phi+(E-V)\phi=0$$

Here V represents potential energy, E is the constant total energy

Put $Q=e^{i\nu t}\psi(x, y, z)$, thus one gets

$\psi=e^{-i\nu t}Q$, $-\frac{h}{i}\frac{\partial\psi}{\partial t}=h\nu\psi$ and hence the quantification

$$(15) \qquad \frac{h^2}{2m}\Delta\psi-\frac{h}{i}\frac{\partial\psi}{\partial t}-V\psi=0 \quad \text{if } E=h\nu$$

Thus the kinematic device of introducing a field ensures quantification through Schrodinger's equation

Take again Schrodinger's equation in the notation of (13)

$$\Delta W+W\Omega=0$$

In order to get a periodic solution of this we put

$$W=e^{i\nu t}\ \omega(r, y, z) \qquad \text{and hence}$$

$$\Delta W-\frac{\Omega}{r^2}\frac{\partial^2 W}{\partial t^2}=0 \qquad \text{and from (14)}$$

$$\Delta W-\frac{1}{C^2}\frac{\partial^2 W}{\partial t^2}=0, \qquad C=\frac{E}{\sqrt{2m(E-V)}}$$

This shows the geometrical synthesis of wave, corpuscular and quantum theory

Here we may interpolate a remark due to Buhl, about the analytical counter part of geometrical homogeneity of space In fact the fundamental principle of quantum mechanics, the non-commutative nature of operators in quantum space, is simply represented by

homogeneous differential operators At the start we have Euler's theorem

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=mf$$

which associates the operators

$$\frac{\partial}{\partial x} \quad \frac{\partial}{\partial y} \quad \frac{\partial}{\partial z}$$

with $\quad x, \quad y, \quad z$

This gives the combination already mentioned at the beginning of this section

$$\frac{\partial}{\partial x_j}(x_i f)-x_i\frac{\partial}{\partial x_j}f=f$$

The theory of groups indicated previously with the help of Euler's theorem leads to

$$\frac{\partial}{\partial x_j}(x_i f)-x_i\frac{\partial}{\partial x_j}f=0, \quad i\neq j$$
$$=nf, \quad i=j$$

Briefly homogeneity of space means non-commutativity of operators

The expression $\iint_\sigma\left(\frac{dw}{dn}+u\frac{dv}{dn}\right)d\sigma$ which is a transform of the double integral of (10) with the help of Jacobi symbols indicates a certain probability for the presence of a phenomenon in a bundle of canals intercepted by the diaphragm σ In order that the probability for the presence of a similar phenomenon in the volume element $d\tau$ may be expressed analogously we utilise the function $\omega\ (u, v, w)d\tau$. If v be the solution of the Schrodinger equation, the probability is given by $\bar{\omega}(v)d\tau$ We may reduce this simply to $v^2 d\tau$ on considerations regarding simplicity In order that the imaginary solutions may be included we replace $v^2 d\tau$ by $\psi\bar{\psi}\,d\tau$ The study of this expression and its integrals is a limited case of the study of indeterminate conjugates of Hermitian type

$a_{ik}\bar{x}_i x_k$, $a_{ki}=\bar{a}_{ik}$, these are forms constituting a unitary geometry developed by Weyl and Cartan

The considerations briefly sketched bear out the truth of Lie's remark referred to at the start The paper may be concluded by a final reference to a remark of Bateman, which sums up the geometrising of physics by saying that all the fundamental equations of physics may be traced to

$$\nabla\Phi - \frac{\partial \overline{\Phi}_b}{\partial x_i} = 0$$

The whole of theoretical physics is thus a system of transformations in a linear group-space

* Read at the Indian Science Congress (Calcutta Session, Section A, Physics and Mathematics, 7th January, 1935)

Bull. Cal Math. Soc., Vol. XXVI, No. 2 (1934)

6

On certain definite integrals involving Bessel Functions of order zero

BY

S C Mitra

In a previous paper,* I have obtained a number of definite integrals with the help of the Operational Calculus The object of the present paper is to evaluate a number of definite integrals involving Bessel Functions of order zero As frequent references of the previous paper will be necessary, we shall refer to it by C It is believed that most of the results are new

1 We have obtained the formula, (C(3)),

$$\int_0^1 P_n(1-2y^2)e^{-2yz}dy = I_{n+\frac{1}{2}}(z)\, K_{n+\frac{1}{2}}(z), \qquad \ldots \quad (1)$$

($n=a$ positive integer).

Therefore the integral,

$$I=\int_0^\infty I_{n+\frac{1}{2}}(z)\ K_{n+\frac{1}{2}}(z) \sin 2kzdz$$

$$=\int_0^\infty\int_0^1 P_n(1-2y^2)e^{-2yz} \sin 2kz\, dydz.$$

Changing the order of integration, which is permissible, we get

$$I=\frac{1}{2}\int_0^1 \frac{P_n(1-2y^2)ydy}{y^2+k^2}$$

$$=\frac{1}{2}\ Q_n(1+2k^2) \qquad \ldots \quad (2)$$

* *Bull Cal Math Soc*, Vol XXV, No 4, 1933, pp 185-90.

Let us multiply both sides of (C(28)) by $\sin 2kz$ and integrate between the limits zero and infinity * We get (on writing y for y^2 and k for k^2),

$$\int_0^1 \frac{e^{-2yt}\,dy}{k+y} = \sqrt{\frac{2\pi}{t}}\; e^{-t} \sum_{n=0}^{\infty} (2n+1)Q_n(1+2k)\, I_{n+\frac{1}{2}}(t)(k>0) \quad .. \quad (3)$$

Let $k=\frac{1}{2}p$ We have

$$\int_0^1 \frac{2p\, e^{-2yt}\,dy}{p+2y} = \sqrt{\frac{2\pi}{t}}\; e^{-t} \sum_{n=0}^{\infty} (2n+1)pQ_n(1+p)I_{n+\frac{1}{2}}(t)$$

On interpretation, we get by (C(11)),

$$\int_0^1 e^{-2y(t+z)}\,dy = \frac{\pi}{2}\; \frac{e^{-(t+z)}}{\sqrt{tz}} \sum_{n=0}^{\infty} (2n+1)\, I_{n+\frac{1}{2}}(t) I_{n+\frac{1}{2}}(z) \qquad (4)$$

or

$$\frac{\sinh (t+z)}{t+z} = \frac{\pi}{2\sqrt{tz}} \sum_{n=0}^{\infty} (2n+1)\, I_{n+\frac{1}{2}}(t) I_{n+\frac{1}{2}}(z) \quad . \quad (5)$$

Again putting $t=\dfrac{1}{p}$ in (3), we get

$$\int_0^1 \frac{J_0(yz)y\,dy}{k^2+y^2} = \frac{2}{z} \sum_{n=0}^{\infty} (2n+1)Q_n(1+2k^2)J_{2n+1}(z) \qquad (6)$$

Let us put $k^2=\frac{1}{2}p$ in the above and multiply both sides by p We get by (C(12))

$$\int_0^1 e^{-2y^2t} J_0(yz)y\,dy = \sqrt{\frac{\pi}{2t}}\; \frac{e^{-t}}{z} \sum_{n=0}^{\infty} (2n+1)\, I_{n+\frac{1}{2}}(z) J_{2n+1}(z) \quad ... \quad (7)$$

In (7) let us put $t=\dfrac{1}{p}$ and interpret We get

$$\int_0^1 J_0(yt)J_0(yz)\,y\,dy = \frac{2}{tz} \sum_{n=0}^{\infty} (2n+1)J_{2n+1}(t)J_{2n+1}(z) \quad ... \quad (8)$$

* In this and in subsequent cases, the process is justifiable

Again in C(25), C(26) and C(28), let us put

$$t=\frac{1}{p} \quad \text{Remembering that } \sqrt{p}\, e^{-\frac{1}{p}} I_v\left(\frac{1}{p}\right) = \frac{J_v(2\sqrt{2x})}{\sqrt{\pi x}},$$

we get

$$\int_0^1 J_0(yt) \sin 2yz dy = \frac{\pi}{t} \sum_{n=0}^{\infty} (2n+1) J^2_{n+\frac{1}{2}}(z) J_{2n+1}(t) \tag{9}$$

$$\int_0^1 J_0(yt) \cos 2yz dy = \frac{\pi}{t} \sum_{n=0}^{\infty} (-1)^n (2n+1) J_{n+\frac{1}{2}}(z) J_{-n-\frac{1}{2}}(z) \times J_{2n+1}(t), \tag{10}$$

and

$$\int_0^1 J_0(yt) e^{-2yz} dy = \frac{2}{t} \sum_{n=0}^{\infty} (2n+1) I_{n+\frac{1}{2}}(z) K_{n+\frac{1}{2}}(z) J_{2n+1}(t) \tag{11}$$

2 Let us now differentiate both sides of C (20) with respect to t * We get

$$\int_0^1 \sin 2yz \cos 2yt\, dy = \frac{\pi^2}{4} \sum_{n=0}^{\infty} (2n+1) J^2_{n+\frac{1}{2}}(z) \frac{d}{dt} J^2_{n+\frac{1}{2}}(t) \quad \dots \tag{12}$$

Divide both sides by $\sqrt{t}$ and integrate between zero and infinity We have on simplification,

$$\int_0^1 \sin 2x^2 z\, dx = \frac{\pi\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} \sum_{n=0}^{\infty} \frac{2n+1}{4n+3} \frac{\Gamma(n+\frac{1}{4})}{\Gamma(n+\frac{3}{4})} J^2_{n+\frac{1}{2}}(z) \quad \cdot\cdot \tag{13}$$

Again dividing both sides by $\sqrt{z}$ and integrating as before (with respect to z) between zero and infinity, we get

$$\int_0^1 \cos 2z^2 t dz = \frac{\pi}{2} \frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})} \sum_{n=0}^{\infty} \frac{2n+1}{4n+1} \frac{\Gamma(n+\frac{3}{4})}{\Gamma(n+\frac{1}{4})} \frac{d}{dt} J^2_{n+\frac{1}{2}}(t) \tag{14}$$

* The differentiation is justifiable.

From (14) it is not difficult to deduce

$$\int_0^1 \frac{\sin 2z^2 t}{z^2}\, dz = \frac{\pi\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})} \sum_{n=0}^{\infty} \frac{2n+1}{4n+1} \frac{\Gamma(n+\frac{3}{4})}{\Gamma(n+\frac{1}{4})} J^2_{n+\frac{1}{2}}(t) \qquad (15)$$

Let us now multiply both sides of (15) by e^{-at} and integrate between zero and infinite We get

$$\int_0^1 \frac{dz}{a^2+z^4} = 2\, \frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})} \sum_{n=3}^{\infty} \frac{2n+1}{4n+2} \frac{\Gamma(n+\frac{3}{4})}{\Gamma(n+\frac{1}{4})} Q_n(1+2a^2) \quad \cdots \quad (16)$$

Let $a^2 = \frac{1}{2}p$. Then

$$\int_0^1 \frac{p\,dz}{p+2z^4} = \frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})} \sum_{n=0}^{\infty} \frac{2n+1}{4n+1} \frac{\Gamma(n+\frac{3}{4})}{\Gamma(n+\frac{1}{4})} pQ_n(p+1)$$

on interpretation, we have, by C (11),

$$\int_0^1 e^{-2z^4 t}\, dz = \frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})} \sqrt{\frac{\pi}{2t}} e^{-t} \sum_{n=0}^{\infty} \frac{2n+1}{4n+1} \frac{\Gamma(n+\frac{3}{4})}{\Gamma(n+\frac{1}{4})} I_{n+\frac{1}{2}}(t) \quad \cdot\cdot \quad (17)$$

Again putting $t=\frac{1}{p}$ in the above and interpreting we get

$$\int_0^1 J_0(z^2 t)\,dz = 2\, \frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})} \sum_{n=0}^{\infty} \frac{2n+1}{4n+2} \frac{\Gamma(n+\frac{3}{4})}{\Gamma(n+\frac{1}{4})} \frac{J_{2n+1}(t)}{t} \quad \cdot\cdot \quad (18)$$

In a similar manner, we can also deduce that

$$\int_0^1 z^2 e^{-2z^4 t}\, dz = \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} \sqrt{\frac{\pi}{2t}} e^{-t} \sum_{n=0}^{\infty} \frac{2n+1}{4n+3} \frac{\Gamma(n+\frac{1}{4})}{\Gamma(n+\frac{3}{4})} I_{n+\frac{1}{2}}(t), \quad \cdot \quad (19)$$

and

$$\int_0^1 z^2 J_0(z^2 t)\,dz = 2\, \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} \sum_{n=0}^{\infty} \frac{2n+1}{4n+3} \frac{\Gamma(n+\frac{1}{4})}{\Gamma(n+\frac{3}{4})} \frac{J_{2n+1}(t)}{t} \quad \cdots \quad (20)$$

3 Let us now consider (8) Let us multiply both sides by e^{-z} and e^{-at} and integrate in succession with respect to z and t between zero and infinity We get

$$\int_0^1 \frac{ydy}{a^2+y^2}=2\sum_{n=0}^{\infty}\frac{1}{(2n+1)}(\sqrt{a^2+1}-a)^{4n+2}. \qquad (21)$$

Let $a=p$. Then

$$\int_0^1 \frac{pydy}{p^2+y^2}=2\sum_{n=0}^{\infty}\frac{p}{2n+1}(\sqrt{p^2+1}-p)^{4n+2}$$

on interpretation, this leads to

$$\int_0^1 \sin yt\,dy=4\sum_{n=0}^{\infty}\frac{J_{4n+2}(t)}{t}$$

or

$$\frac{1-\cos t}{4}=\sum_{n=0}^{\infty}J_{4n+2}(t) \qquad (22)$$

a result which can be deduced from an expansion given by Joliffe.*

Now let us put $t=\frac{1}{p}$ in (23) We have, since

$$J_\nu\left(\frac{1}{p}\right)=J_\nu(\sqrt{2t})I_\nu(\sqrt{2t})\ \dagger$$

$$\tfrac{1}{4}(1-\operatorname{ber}(2\sqrt{2}\ x)=\sum_{n=0}^{\infty}J_{4n+2}(2x)I_{4n+2}(2x) \qquad (23)$$

where we have written $2x$ for $\sqrt{2t}$

* Watson, *Bessel Functions*, p 528

† *Bull Cal Math Soc*., Vol XXV, No. 1933, p. 83

We have proved elsewhere * that

$$\operatorname{ber}(2\sqrt{2x}) = J_0(2x)I_0(2x) - 2J_2(2x)I_2(2x) + 2J_4(2x)I_4(2x) - \ldots \qquad (24)$$

Therefore taking into account, the equation (23) we get

$$\tfrac{1}{2}(1+\operatorname{ber}(2\sqrt{2}x) = J_0(2x)I_0(2x) + 2J_4(2x)I_4(2x) + 2J_8(2x)I_8(2x) + \qquad (25)$$

Bull. Cal Math Soc, Vol. XXVI, No 2 (1934).

* *loc cit*, p 98.

7

On the Product of Parabolic Cylinder Functions

BY

S C Dhar

1 *Introduction.*—In a recent paper * Dr Mitra has given a method by means of which he has obtained an expression for the squares of parabolic cylinder functions The method he has employed, is that of the transformation of the differential equation and its solution by series, a method used by Appell† and Heine‡ in Harmonic Functions It is the purpose of this paper to obtain expressions for the product and squares of parabolic cylinder functions by contour integration and also to obtain an *Addition Theorem* corresponding to the Neumann Addition Theorem in Bessel's functions

2. The well-known differential eqnation for the parabolic cylinder functions is given by

$$\frac{d^2y}{dz^2}+(n+\tfrac{1}{2}-\tfrac{1}{4}z^2)y=0. \qquad \ldots \quad (1)$$

The solution of the above equation $D_n(z)$ is given in the contour form §

$$D_n(z)=-\frac{\Gamma(n+1)}{2\pi i}e^{-\frac{1}{4}z^2}\int_{\infty}^{(0+)}e^{-zt-\frac{1}{2}t^2}(-t)-n-1\,dt, \quad \ldots \quad (2)$$

which reduces, when n is an *integer* to the form

$$D_n(z)=-\frac{n!}{2\pi i}e^{-\frac{1}{4}z^2}\int^{(0+)}e^{-zt-\frac{1}{2}t^2}(-t)^{-n-1}dt \qquad \ldots \quad (3)$$

* S C. Mitra *Proc Edinburgh Math Soc* (2) 4 (1934) 27-32

† Appell. *Competes Rendus*, 91

‡ Heine *Kugelfunctionen*, 1878

§ Whittaker and Wotson *Modern Analysis*, (1927) 349-350

In this we make the substitution $t=-e^{-i\theta}$ and obtain

$$D_n(z)=\frac{n!}{2\pi}e^{-\frac{1}{4}z^2}\int_{-\pi}^{\pi}e^{z\cos\theta-\frac{1}{2}\cos 2\theta+i(n\theta-z\sin\theta+\frac{1}{2}\sin 2\theta)}\,d\theta \tag{4}$$

which can be reduced into the form

$$D_n(z)=\frac{n!}{2\pi}\;e^{-\frac{1}{4}z^2}\int_0^{\pi}e^{z\cos\theta-\frac{1}{2}\cos 2\theta}$$
$$\times\cos(n\theta-z\sin\theta+\tfrac{1}{2}\sin 2\theta)d\theta \quad . \tag{5}$$

3 Let us consider the product $D_n(z)$ $D_m(z)$ where both n and m are positive integers and apply (4) We get

$$D_n(z)D_m(z)=\frac{n!m!}{4\pi^2}\;e^{-\frac{1}{2}z^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}e^{z(\cos\theta+\cos\phi)-\frac{1}{2}(\cos 2\theta}$$

$$+\cos 2\phi)\;e^{i\{n\theta+m\phi-z(\sin\theta+\sin\phi)+\frac{1}{2}(\sin 2\theta+\sin 2\phi)\}}\,d\theta d\phi$$

Put

$$\theta-\phi=2\alpha$$

$$\theta+\phi=2\beta$$

and the expression reduces to

$$D_n(z)D_m(z)=\frac{n!m!}{4\pi^2}\;e^{-\frac{1}{2}z^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}e^{2z\cos\alpha e^{-i\beta}-\cos 2\alpha e^{-2i\beta}}$$

$$\times e^{i(n-m)\alpha+i(n+m)\beta}d\alpha d\beta \quad \ldots \tag{6}$$

Now, put $t=-2\cos\alpha e^{-i\beta}$ and the expression (6) becomes

$$D_n(z)D_m(z)=\frac{n!m!}{2\pi}\;e^{-\frac{1}{4}z^2}\int_{-\pi}^{\pi}\;e^{i(n-m)\alpha}d\alpha,$$

$$\Sigma'\frac{(2\cos\alpha)^{n+m-p}}{p!(n+m-2p)!}\,D_{n+m-2p}(z),$$

where Σ means summation from $p=0$ to $p=\frac{1}{2}(n+m)$ or $\frac{1}{2}(n+m-1)$ according as $(n+m)$ is *even* or *odd*, for we know that

$$\int^{(0+)} e^{-zt-\frac{1}{2}t^2-\frac{1}{4}z^2}(-t)^{y-1}dt=0, \text{ when } \mathrm{R}(y)>0$$

The above expression can now be written in the form

$$\mathrm{D}_n(z)\mathrm{D}_m(z)$$

$$=\frac{n!m!}{\pi}\ e^{-\frac{1}{4}z^2}\ \Sigma'\ \frac{2^{n+m-2p+1}}{p!(n+m-2p)!}\mathrm{D}_{n+m-2p}(z)$$

$$\times\int_0^{\frac{\pi}{2}} \cos(n-m)a \cos^{n+m-2p}a\,da$$

But, by Cauchy's integral

$$\int_0^{\frac{\pi}{2}} \cos^{q+r-2}\theta \cos(q-r)\theta d\theta=\frac{\pi}{(q+r-1)2^{q+r-1}\mathrm{B}(q,r)}$$

*Hence $\mathrm{D}_n(z)\mathrm{D}_m(z)=\mathrm{D}_0(z)\sum_{p=0}^{p=m}\frac{n!m!}{p!(n-p)!(m-p)!}\mathrm{D}_{n+m-2p}(z)$ (7)

4 In (6) we make the substitution,

$$t=-\sqrt{2}\cos a\, e^{-i\beta}$$

and get

$$\mathrm{D}_n(z)\mathrm{D}_m(z)$$

$$=-\frac{n!\,m!}{4i\pi^2}\int_{-\pi}^{\pi} e^{i(n-m)a}(\sqrt{2}\cos a)^{n+m}da$$

$$\times\int^{(0+)} e^{-\sqrt{2}zt-\frac{1}{2}t^2-\frac{1}{4}z^2+\frac{1}{2}(\tan^2 a)t^2}(-t)^{-n-m-1}dt$$

$$=\frac{n!m!}{\pi}\sum_p{}'\frac{(\sqrt{2})^{n+m-2p+2}}{p!(n+m-2p)!}\mathrm{D}_{n+m-2p}(z\sqrt{2})$$

* *Cf* Gorak Prasad. *Proc. Benares Math Soc*, 2 (1920) 7

$$\times \int_0^{\frac{\pi}{2}} \cos (n-m)\, a \sin^{2p} a \cos^{n+m-2p} a \, da$$

$$= n!\, m! \sum_p{}' \frac{(\sqrt{2})^{n+m-2p+2}}{p!(n+m-2p)!} B_{n+m-2p} D_{n+m-2p}(2\sqrt{2}) \quad .. \quad (8)^*$$

where $\sum'$ has the usual meaning and where

$$B_{n+m-2p} = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} \cos (n-m)\, a \sin^{2p} a \cos^{n+m-2p} a \, da.$$

5. It is now easy to obtain expressions for the squares of parabolic cylinder functions and certain integrals connected with them.

In (7) make $n=m$, and we get

$$D_n^2(z) = D_0(z) \sum_{p=0}^{n} \frac{(n!)^2}{p!\{(n-p)!\}^2} D_{2n-2p}(z)$$

$$= D_0(z) \sum_{m=0}^{n} \frac{(n!)^2}{(m!)^2(n-m)!} D_{2m}(z) \quad (9)$$

Multiply both sides by $D_{2m}(z)$ and integrate between $-\infty$ and $+\infty$, we get

$$\int_{-\infty}^{\infty} e^{\frac{1}{4}z^2} D_n^2(z) D_{2m}(z) dz = \frac{(n!)^2(2m)!}{(m!)^4(n-m)!} \sqrt{2\pi}, \ (n \geq m) \quad . \quad (10)$$

Again from (9) we get

$$\int_0^{\infty} D_n^4(z) dz = \sum_{m=0}^{n} \frac{(n!)^2}{(m!)^2(n-m)!} \int_0^{\infty} e^{-\frac{1}{4}z^2} D_n^2(z) D_{2m}(z) dz \dagger$$

$$= \sqrt{\pi}\, n! \frac{1 \cdot 3 \ldots (2n-1)}{2^{n+1}} + \sum_{m=1}^{n} \sqrt{\pi} \frac{(n!)^2(2m!)^2(2n-2m-1)!(n-m)}{\{(n-m)!\}^2(m!)^4 2^{2n+2m}} \quad (11)$$

* Cf. S C Mitra *Bull Calcutta Math Soc*, 17 (1926)

† S C Mitra *Proc Edinburgh Math Soc*, loc cit, 31

Again in (8) we put $n=m$, and we get

$$D_n^2(z)=(n!)^2 \sum_{p=0}^{n} \frac{(\sqrt{2})^{2n-2p+2}}{p!(2n-2p)!} D_{2n-2p}(z\sqrt{2}) \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \sin^{2p}a \cos^{2n-2p}a\,da$$

$$=\frac{1}{2^n} \sum_{p=0}^{n} \frac{n!(2p-1)(2p-3)\,..3\,1}{(p!)(n-p)!} D_{2n-2p}\ (z\sqrt{2})\ * \qquad \ldots \quad (12)$$

6 We will proceed to obtain expressions for the product of two parabolic cylinder functions with different arguments For this we take Adamaff's † integral in the form

$$D_n(x)=(-1)^{\frac{1}{2}n}2^{n+1}(2\pi)^{-\frac{1}{2}}e^{\frac{1}{4}x^2}\int_{-\infty}^{\infty} u^n e^{-2u^2+2ixu}du \qquad .. \quad (13)$$

Hence

$$D_n(x)D_m(x)=(-1)^{\frac{1}{2}(m+n)}\frac{2^{n+m+2}}{2\pi}$$

$$\times\, e^{\frac{1}{4}(\chi^2+x^2)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u^n t^m e^{-2u^2-2t^2+2i\chi u+2ixt}\,du dt$$

Make the substitution

$$u-t=2T$$

$$u+t=2U,$$

and the expression reduces to

$$D_n(\chi)D_m(x)=(-1)^{\frac{1}{2}(m+n)}\frac{2^{m+n+2}}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{1}{8}\{(\chi+x)^2+(\chi-x)^2\}}$$

$$\times\,(U+T)^n(U-T)^m e^{-4T^2-4U^2+2i(\chi+x)U+2i(\chi-u)T}\,dUdT$$

From the known expansion

$$(U+T)^n(U-T)^m=\sum_{r=0}^{m+n} A_{n+m-r}U^{n+m-r}T^r,$$

* Ditto Ditto

† Adamoff · *Aun. de l'ttustitul poletechugue de St Petersburg*, 5 (1906) 127-43.

where

$$A_{n+m-r}=\frac{n(n-1)\quad(n-r+1)}{r!}\,F(-r,-m\,,\,n-r+1\,,-1)$$

we obtain

$$D_n(\chi)\,D_m(z)=\frac{1}{(\sqrt{2})^{m+n}}\sum_{r=0}^{n+m}A_{n+m-r}D_{n+m-r}\left(\frac{\chi+x}{\sqrt{2}}\right)D_r\left(\frac{\chi-\iota}{\sqrt{2}}\right)\quad(14)$$

If, in this, we make $\chi=x=z$, the expression (14) reduces to the from (8)

The expression (14) is now transformed by the substitution

$$\chi+x=z_1\sqrt{2}$$

$$\chi-x=z_2\sqrt{2}$$

into the form

$$D_n\left(\frac{z_1+z_2}{\sqrt{2}}\right)D_m\left(\frac{z_1+z_2}{\sqrt{2}}\right)=\frac{1}{(\sqrt{2})^{m+n}}\sum_{r=0}^{m+n}A_{n+m-r}D_{n+m-r}(z_1)D_r(z_2)\quad(15)$$

whence we can obtain the integral

$$D_r(z\sqrt{2})=\frac{(\sqrt{2})^{m+n-1}}{(n+m-r)!\,A_{n+m-r}}$$

$$\times\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}D_n(a+z)D_m(a-z)D_{n+m-r}(a\sqrt{2})\,da\quad\ldots\quad(16)$$

From (15) we can also obtain the integral

$$\frac{1}{2\pi}\int_{-8}^{\infty}\int_{-\infty}^{\infty}D_n\left(\frac{z_1+z_2}{\sqrt{2}}\right)D_n\left(\frac{z_1-z_2}{\sqrt{2}}\right)D_{n+m-r}(z_1)D_r(z_2)\,dz_2\,dz_1$$

$$=\frac{(n+m-r)!\;n(n-1)\ldots(n-m+1)}{(\sqrt{2})^{m+n}}\,F(-r,-m\,,\,n-r+\iota\,;-1)\quad(17)$$

7 We will proceed to obtain the *Addition Theorem*

In (3), we put $z=x \sin a+y \cos a$, and get $D_n(x \sin a+y \cos a)$

$$=-\frac{n!}{2\pi i} e^{-\frac{1}{4}(x \sin a+y \cos a)^2} \int^{(0+)} e^{-(x \sin a+y \cos a)\, t-\frac{1}{2}t^2} (-t)^{-n-1} dt$$

$$=-\frac{n!}{2\pi i} e^{-\frac{1}{4}(y^2 \cos^2 a+2xy \sin a \cos a-n^2 \cos^2 a)} \sum_{m=0}^{\infty} \frac{D_m(x)}{m!} \sin^m a$$

$$\times \int^{(0+)} e^{-y \cos a\, t-\frac{1}{2} \cos^2 a\, t^2} (-t)^{m-n-1} dt,$$

since it is known that

$$e^{zt-\frac{1}{2}t^2-\frac{1}{4}z^2} = \sum_{m=0}^{\infty} \frac{D_m(z)}{m!} t^m$$

Put $t \cos a=u$, and it is transformed into
$D_n(x \sin a+y \cos a)$

$$=-\frac{n!}{2\pi i} e^{-\frac{1}{4}(y^2 \cos^2 a+2xy \cos a \sin a-x^2 \cos^2 a} \sum_{m=0}^{\infty} \frac{D_m(x)}{m!} \sin^m a$$

$$\times \int^{(0+)} e^{-yu-\frac{1}{2}u^2} (-u^{m-n-1} \cos^{n-m} a da$$

Now, we know that

$$\int^{(0+)} e^{-\frac{1}{2}u^2-yu} (-u)^{m-n-1} du=0, \text{ when } m>n$$

Therefore

$D_n(x \sin a+y \cos a)$

$$=e^{\frac{1}{4}(x \cos a-y \sin a)^2} \sum_{m=0}^{n} \frac{D_m(x)}{m!} \frac{D_{n-m}(y)}{(n-m)!} \sin^m a \cos^{n-m} a \tag{18}$$

Now, when we take $a=\frac{\pi}{4}$ the above expression reduces to the form already obtained, * *viz*,

* Dhar and Shastri *Phil Map* (7) 18, (1934), 404
Dhar; *Jour London Math Soc* (to be published by Shastri)

$$D_n(x+y)=\frac{n!}{(\sqrt{2})^n}e^{\frac{1}{4}(x-y)^2}\sum_{m=0}^{n}\frac{D_m(x\sqrt{2})}{m!}\frac{D_{n-m}(y\sqrt{2})}{(n-m)!} \quad .. \quad (19)$$

We can, by the contour-method, find an expression for $D_n(x+y)$ which will hold for all values of n

Take the form (2) and put $z=x+y$, and we get

$$D_n(x+y)=-\frac{\Gamma(n+1)}{2\pi i}e^{-\frac{1}{2}xy}$$

$$\times\int_{\infty}^{(+0)} e^{-xt-\frac{1}{2}t^2-\frac{1}{4}x^2}(-t)^{-n-1}e^{\frac{1}{2}t^2}\sum_{m=0}^{\infty}\frac{D_m(y)}{m!}(-t)^m dt$$

Expanding $e^{\frac{1}{2}t^2}\sum\frac{D_m(y)}{m!}(-t)^m$ and simplipying, we get

$$D_n(x+y)=\Gamma(n+1)\,e^{-\frac{1}{2}xy}\sum_{r=0}^{\infty}\frac{D_{n-r}(x)}{\Gamma(n-r+1)}\left\{\frac{D_r(y)}{r!}+\frac{1}{2}\frac{D_{r-2}(y)}{(r-2)!}\right.$$

$$\left.+\frac{1}{2^2}\frac{D_{r-4}(y)}{(r-4)!\,2!}+\frac{1}{2^3}\frac{D_{r-6}(y)}{(r-6)!\,3!}+\dots\right\} \quad . \quad (20)$$

8

Note on some two dimensional problems of elasticity connected with plates having triangular boundaries

BY

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Introduction

Besides the torsion problem,* there are very few problems of elasticity connected with triangular boundaries of which the exact solution is known It is, however, found that in a few problems concerning thin plates with triangular boundaries, trilinear co-ordinates can be used with much advantage, while the introduction of other co-ordinates leads to complication To demonstrate the use of trilinear co-ordinates in some such problems, the present note is written, and for this purpose the following simple cases have been considered, *viz* —

(1) Bending under normal pressure of a thin plate in the form of an equilateral triangle with supported edge,

(2) Transverse vibration of a plate in the form of an equilateral triangle with supported edge,

(3) Buckling under uniform edge thrust of a plate in the form of an equilateral triangle with supported edge

It is believed that the method employed in these problems is new and that the results deduced, have not been previously obtained by any writer

* Vide '*The Mathematical Theory of Elasticity*' by A. E. H. Love, 4th edition, pp 319 and 320

(1) *Bending under normal pressure of a thin plate in the form of an equilateral triangle*

Let the equilateral triangle ABC be the midsection of the plate of which the thickness is $2h$ We take the origin at the point 0 which is the centroid of the triangle The axis $0x$ is taken perpendicular to the side BC and the axis $0y$, parallel to this side

If (x, y) be the Cartesian co-ordinates of a point P within the triangle, p_1, p_2, p_3 three perpendiculars from P on CA, AB and BC respectively, $2a$, the length of each side of the triangle, and r, the radius of the inscribed circle, then we get

$$p_1 = r + \frac{x}{2} - \frac{y\sqrt{3}}{2},$$

$$p_2 = r + \frac{x}{2} + \frac{y\sqrt{3}}{2},$$

$$p_3 = r - x. \tag{1 1}$$

Hence

$$p_1 + p_2 + p_3 = 3r = a\sqrt{3} = k \text{ (say)} \tag{1 2}$$

The differential equation satisfied by the small deflection ω of a thin plate bent by uniform normal pressure z per unit of area is given by

$$\nabla_1^4 \omega = \frac{Z}{D} = z_0^* \text{ (say)} \quad \ldots \tag{1 3}$$

where D is the flexural rigidity, and

$$\nabla_1^2 \text{ stands for } \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

which expressed in terms of p_1, p_2, p_3, becomes

$$\frac{\partial^2}{\partial p_1^2} + \frac{\partial^2}{\partial p_2^2} + \frac{\partial^2}{\partial p_3^2} - \frac{\partial^2}{\partial p_1 \partial p_2} - \frac{\partial^2}{\partial p_2 \partial p_3} - \frac{\partial^2}{\partial p_3 \partial p_1}. \quad \ldots \tag{1 4}$$

* Love's '*Elasticity*' 4th edition, p 488.

At the supported edges, $p_1=0$, $p_2=0$ and $p_3=0$, we must have

$$\omega=0, \text{ and}$$

$$\frac{\partial^2\omega}{\partial\nu^2}+\sigma\left(\frac{\partial^2\omega}{\partial s^2}+\frac{1}{\rho}\frac{\partial\omega}{\partial\nu}\right)^{*}=0 \quad \ldots \quad (1.5)$$

Since

$$\nabla_1^2\omega=\frac{\partial^2\omega}{\partial\nu^2}+\frac{\partial^2\omega}{\partial s^2}+\frac{1}{\rho}\frac{\partial\omega}{\partial\nu},$$

the last condition becomes

$$\sigma\nabla_1^2\omega+(1-\sigma)\frac{\partial^2\omega}{\partial\nu^2}=0 \quad . \quad (1.6)$$

where σ is Poisson's ratio and $d\nu$ an element of outward drawn normal at a point on the boundary

As a solution of the equation (1.3), we assume

$$\omega=A(p_1^3p_2p_3+p_2^3p_1p_3+p_3^3p_1p_2)+Bp_1p_2p_3, \quad (1.7)$$

A and B being constants

Performing the operation given in (1.4), we find by virtue of the relation (1.2)

$$\nabla_1^2\omega=24Ap_1p_2p_3-Ak^3-Bk \quad \ldots \quad (1.8)$$

Again performing the operation (1 4) on the above expression, it will be seen that the equation (1 3) is satisfied if

$$A=-\frac{z_0}{24k}. \quad \ldots \quad (1.9)$$

* Love's *Elasticity*, 4th edition, p. 296.

It is apparent from (1.7) that $\omega=0$ on the boundaries $p_1=0$, $p_2=0$ and $p_3=0$.

Corresponding to the boundaries $p_1=0$, $p_2=0$ and $p_3=0$, we have

$$\frac{\partial}{\partial \nu}=-\frac{\partial}{\partial p_1}+\frac{1}{2}\frac{\partial}{\partial p_2}+\frac{1}{2}\frac{\partial}{\partial p_3}, \qquad \dots \quad (1.10)$$

$$\frac{\partial}{\partial \nu}=-\frac{\partial}{\partial p_2}+\frac{1}{2}\frac{\partial}{\partial p_3}+\frac{1}{2}\frac{\partial}{\partial p_1}, \text{ and} \qquad \dots \quad (1.11)$$

$$\frac{\partial}{\partial \nu}=-\frac{\partial}{\partial p_3}+\frac{1}{2}\frac{\partial}{\partial p_1}+\frac{1}{2}\frac{\partial}{\partial p_2} \qquad \dots \quad (1.12)$$

respectively

Performing the operation (1.10) twice on ω in succession and then putting $p_1=0$, we find for this boundary

$$\frac{\partial^2\omega}{\partial \nu^2}=-Ak^3-Bk \qquad \dots \quad (1.13)$$

Hence it can be seen from (1.8) and (1.13) that

$$\sigma\nabla_1^2\omega+(1-\sigma)\frac{\partial^2\omega}{\partial \nu^2}=0 \text{ when } p_1=0,$$

if $\quad B=-Ak^2=\dfrac{z_0 k}{24}$. ... (1.14)

Similarly obtaining $\dfrac{\partial^2\omega}{\partial \nu^2}$ for other boundaries, we find that the condition (1.6) is satisfied on them for the same value of B as obtained above.

Hence the required value of

$$\omega=-\frac{z_0}{24k}\left[p_1^3p_2p_3+p_2^3p_1p_3+p_3^3p_1p_2\right]+\frac{zok}{24}p_1p_2p_3 \quad . \quad (1\ 15)$$

At the origin $p_1=p_2=p_3=r$, and there the deflection

$$\omega=\frac{zr^4}{12D}\ .$$

(2) *Transverse vibration of a thin plate in the form of an equilateral triangle with supported edge*

The equation of transverse vibration of a plate of density ρ and thickness $2h$ is

$$\nabla_1^4\omega=-\frac{2\rho h}{D}\ \frac{d^2\omega}{dt^2}\ ^*$$

where $\quad . \quad (2.1)$

$$D=\frac{2}{3}\ \frac{Eh^3}{1-\sigma^2}.$$

When the plate vibrates in a normal mode, ω is of the form $W\cos(pt+t)$

Whence we find that W satisfies the equation

$$\nabla_1^4W=\frac{3p(1-\sigma^2)p^2}{Eh^2}\ W. \quad .. \quad (2\ 2)$$

For a supported edge, we have as in the former problem $W=0$, and

$$\sigma\nabla_1^2W+(1-\sigma)\frac{\partial^2W}{\partial\nu^2}=0 \quad ... \quad (2\ 3)$$

at any point of the boundary.

* Love's Elasticity, 4th edition, p 496

For a simple type of vibration, we can write

$$W=A_m \left[\sin\frac{2m\pi p_1}{k}+\sin\frac{2m\pi p_2}{k}+\sin\frac{2m\pi p_3}{k}\right], \qquad (2\cdot4)$$

where m is an integer and A_m, a constant

When $p_1=0$,

$$W=A_m \left[\sin\frac{2m\pi p_2}{k}+\sin\frac{2m\pi p_3}{k}\right]$$

$$=2A_m \sin\frac{m\pi}{k}(p_2+p_3)\cos\frac{m\pi}{k}(p_2-p_3)$$

$$=0$$

(since $p_2+p_3=k$ when $p_1=0$)

Similarly we find that $W=0$ on the other sides.

Again

$$\nabla_1^2 W=-\frac{4m^2\pi^2}{k^2} \qquad (2\cdot5)$$

hence $\nabla_1^2 W=0$ on the three bounding lines.

At any point on the line C A

$$\frac{\partial W}{\partial \nu}=-\frac{\partial W}{\partial p_1}+\frac{1}{2}\frac{\partial W}{\partial p_2}+\frac{1}{2}\frac{\partial W}{\partial p_3}$$

$$=\frac{2m\pi}{k}A_m\left[-\cos\frac{2m\pi p_1}{k}+\tfrac{1}{2}\cos\frac{2m\pi p_2}{k}+\tfrac{1}{2}\cos\frac{2m\pi p_3}{k}\right]$$

and $$\frac{\partial^2 W}{\partial \nu^2}=-\frac{4m^2\pi^2}{k^2}A_m\left[+\sin\frac{2m\pi p_1}{k}+\tfrac{1}{4}\sin\frac{2m\pi p_2}{k}+\tfrac{1}{4}\sin\frac{2m\pi p_3}{k}\right]$$

when $p_1=0$, the r. h. s

$$=-\frac{2m^2\pi^2}{k^2}\sin\frac{m\pi}{k}(p_2+p_3)\cos\frac{m\pi}{k}(p_2-p_3)$$

$$=0.$$

Thus we find that

$$\sigma\nabla_1^2 W+(1-\sigma)\frac{\partial^2 W}{\partial\nu^2}=0 \text{ when } p_1=0$$

In a similar way it can be shown that the boundary condition (2 3) holds good on the other lines $p_2=0$ and $p_3=0$

The expression for W given in (2 4) satisfies the differential equation (2.2), if

$$\frac{16m^4\pi^4}{k^4}=\frac{3\rho(1-\sigma^2)}{Eh^2}p^2$$

i e, if

$$p^2=\frac{16Eh^2\pi^4m^4}{3\rho(1-\sigma^2)k^4}, \qquad \text{.. (2 6)}$$

For different integral values of m, this result gives the periods of vibration of the type assumed

(3) *Buckling under edge thrust of a plate in the form of an equilateral triangle*

Let ω be the small deflection due to the uniform edge thrust P per unit of area of the rim surface Then the differential equation satisfied by ω is *

$$D\nabla_1^4\omega=-2hP\nabla_1^2\omega,$$

or

$$\nabla_1^4\omega+\lambda^2\nabla_1^2\omega=0, \qquad \text{... (3 1)}$$

where

$$\lambda^2=\frac{3(1-\sigma^2)P}{Eh^2}. \qquad \text{... (3 2)}$$

At the supported edge we shall have

$\omega=0$ and

$$\sigma\nabla_1^2\omega+(1-\sigma)\frac{\partial^2\omega}{\partial\nu^2}=0 \qquad \text{.. (3 3)}$$

* *Vide* Love's Elasticity, p. 538.

Let us assume

$$\omega = A_m \left[\sin \frac{2m\pi p_1}{k} + \sin \frac{2m\pi p_2}{k} + \sin \frac{2m\pi p_3}{k} \right], \quad \text{..} \quad (3\ 4)$$

where m is an integer and A_m, a constant

We have already seen in the previous section that this value of ω satisfies the boundary conditions stated in (3 3)

Substituting the value of w in the equation (3 1), we get

$$\lambda^2 = \frac{4m^2\pi^2}{k^2}.$$

This gives

$$P = \frac{4Eh^2m^2\pi^2}{3(1-\sigma^2)k^2}. \quad \text{..} \quad (3\ 5)$$

The least value of P consistent with the assumed small displacement ω is therefore

$$= \frac{4Eh^2\pi^2}{3(1-\sigma^2)k^2} \quad \text{...} \quad (3.6)$$

when $m=1$

9

"On an Octavic related to two Co-planar tetrads of Points"

BY

P N. Dasgupta

[*Abstract* The cross-ratios of the two pencils subtended at a point by two tetrads coplanar with the point, if connected by a second degree relation, the locus of the point is an eighth degree curve. It has been shewn that this curve has double points at the two given tetrads and at another tetrad With every point P_1 on the Octavic is associated, under certain conditions, three other tetrads P_2, P_3, P_4 such that $(P_1 P_2)$, $(P_2 P_3)$, $(P_3 P_4)$, $(P_4 P_1)$ are inscribable in a conic with one or other of the given tetrads A generalised result is also given]

Introduction A point when joined to a tetrad of points in its plane, gives a pencil with a cross-ratio This same point when joined to another tetrad in the same plane, gives another cross-ratio, which if connected by a second degree relation with the other cross-ratio, gives a locus for the point which is an eighth degree curve The object of the present paper is to study some geometrical aspects of this curve and its generalised analogue

1 Let (x_r, y_r) where $r=1, 2, 3, 4$ and $5, 6, 7, 8$ give the two tetrads, referred to later as points 1, 2, 3, 4 and 5, 6, 7, 8. If (x, y) be a variable point P, the line joining P to (x_r, y_r) is, $(y-y_r)/(x-x_r)=m_r$, say,

then $m_r-m_s=-\begin{vmatrix} x, & y, & 1 \\ x_r, & y_r, & 1 \\ x_s, & y_s, & 1 \end{vmatrix}=-(rs)$, suppose

Now let $\rho=\{1234\}$

$$\text{then} \quad \rho=\frac{(m_1-m_2)(m_3-m_4)}{(m_1-m_4)(m_3-m_2)}=\frac{(12)(34)}{(14)(32)},$$

$$\text{and } \sigma=\{5678\}=\frac{(56)(78)}{(58)(76)}$$

Now, if it is provided that $a\rho^2+2h\rho\sigma+b\sigma^2+2g\rho+2f\sigma+c=0$ ... (1) we get as locus of P, the eighth degree curve,

$$al^2q^2+2hlmpq+bp^2m^2+2glmq^2+2fpqm^2+cq^2m^2=0 \quad \dots \quad (2)$$

Where

$$l=(12)(34),\ m=(14)(23),$$

$$p=(56)(78) \text{ and } q=(58)(76).$$

Let the curve in (2) be denoted by Γ

2 We can write (2) in the form

$$lq(alq+2hmp+2gmq)+m^2(bp^2+2fpq+cq^2)=0 \quad (3)$$

So, $l=o$ intersects Γ where either

$$m^2=o \text{ or } bp^2+2fpq+cq^2=0 \quad \dots \quad (4)$$

Again $m=o$ intersects the curve where $l^2q^2=0$ ... (5)

Also similarly $q=o$ intersects Γ where $p^2m^2=0$ ... (6)

Γ has double points at each of the points 1, 2, 3, 4 and 5, 6, 7, 8 and also at the following intersections of the lines (14) and (58), (14) and (67), (23) and (67), (23) and (58) ... (7)

3 If a point P is taken on Γ it gives a determinate cross-ratio ρ with the tetrad 1, 2, 3, 4 and another cross-ratio σ with 5, 6, 7, 8. Now P with 1, 2, 3, 4 form a set of points which determine a conic, let us call it $S\rho$ Similarly P with 5, 6, 7, 8 will determine another conic $S\sigma$ say. Now $S\rho$ and $S\sigma$ intersect in four points which are real or in conjugate imaginaries.

We would for the purpose of subsequent work class these simply as 'points' Since P is one of these four points of intersection, there is associated with P, three other points forming a tetrad of points which we denote by $\{\rho, \sigma\}$ Now $S\rho$ intersects Γ in 16 points

of which eight are at 1, 2, 3, 4 which are double points, four given by $\{\rho, \sigma\}$ the remaining four are given by $\{\rho, \sigma'\}$ where σ and σ' are the two roots of (1), considered as a quadratic in σ where ρ has a given value Similarly $S\sigma$ intersects Γ, besides 5, 6, 7, 8 and $\{\rho, \sigma\}$, in four points denoted by $\{\sigma, \rho'\}$ where ρ, ρ' are the roots of (1) considered as a quadratic in ρ, σ having a determinate value

Thus we get, any point P, giving a cross-ratio ρ and σ with the basic tetrads, is associated with three other points forming a tetrad $\{\rho, \sigma\}$, such that associated with it are three other tetrads forming the set,

$$\begin{array}{ll} \{\rho, \sigma\}, & \{\rho', \sigma\} \\ \{\rho, \sigma'\}, & \{\rho', \sigma'\}, \end{array}$$

the components of any one of these being real or in conjugate imaginaries such that any pair of tetrads in a row or in a column is inscribable in a conic with one or other of the basic tetrads 1, 2, 3, 4 and 5, 6, 7, 8

Reciprocally, we have from (1) to (6) If there are in a plane two quadrilaterals with sides 1, 2, 3, 4 and 5, 6, 7, 8 then any line which intersects in groups of points with cross-ratios $\rho = \{1234\}$, $\sigma = \{5678\}$ so that, $a\rho^2 + 2h\rho\sigma + b\sigma^2 + 2g\rho + 2f\sigma + c = 0$, the envelope of the line is the curve C of 8, which has as bitangents

(α) the sides of the quadrilaterals, *viz*, lines

1, 2, 3, 4, 5 8

(β) the joins of the following points —

(14) and (58), (14) and (67), (23) and (67), (23) and (58) .. (9)

And again, any tangent to c with three other lines form a quadrilateral $\{\rho, \sigma\}$ such that associated with it are three other quadrilaterals forming the set,

$$\begin{array}{ll} \{\rho, \sigma\}, & \{\rho', \sigma\} \\ \{\rho, \sigma'\}, & \{\rho', \sigma'\} \end{array}$$

the components of any one of these being real or in conjugate imaginaries, such that any pair of quadrilaterals in a row or column circumscribe a conic along with the sides of one or other of the basic quadrilaterals 1, 2, 3, 4 and 5, 6, 7, 8, ... (10)

4 The results (7) and (8) can be extended to the case where ρ and σ are connected by the general nth degree relation of the type

$$u_n(\rho, \sigma)+u_{n-1}(\rho, \sigma)+ \ldots +u_1(\rho, \sigma)+c=0 \tag{11}$$

where $u_n, u_{n-1}, \ u_1$ represent homogeneous functions of degree n, $n-1$, etc of ρ and σ

It can be shewn that —

(A) the locus of P as from (11) is a curve Σ of the $4n$th degree with n branches through each of the eight points of the two basic tetrads and the four intersections of the lines (14) and (23) with (58) and (67)

(B) Any point on Σ is associated with three other points forming a tetrad $\{\rho, \sigma\}$ such that associated with it are n^2-1 other tetrads, forming the set

$$\begin{array}{lll} \{\rho, \sigma\}, \{\rho_1, \sigma\}, & \ldots & \{\rho_{n-1}, \sigma\} \\ \{\rho, \sigma_1\}, \{\rho_1, \sigma_1\}, & & \{\rho_{n-1}, \sigma_1\} \\ \cdot & \cdots & \\ \{\rho, \sigma_{n-1}\}, \{\rho_1, \sigma_{n-1}\}, & & \{\rho_{n-1}, \sigma_{n-1}\} \end{array}$$

the components of any one of these being real or in conjugate imaginaries such that any group of tetrads, in a row or column is inscribable in a conic with one or other of the basic tetrads 1, 2, 3, 4 and 5, 6, 7, 8, provided that corresponding to ρ, the n values of σ and corresponding to σ, the n values of ρ are real .. (12)

The above result could be reciprocated to give a generalisation of result (10).

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REVIEW

Relativity, Thermodynamics and Cosmology R C Tolman

(Published by Oxford Clarendon Press)

After the classic treatment of the Theory of Relativity by Eddington in his Mathematical Theory of Relativity which till recently was the only serious book in the English language on the subject, there has not for a long time appeared any other book in the same language dealing with the subject with any degree of thoroughness The Theory of Lorentz transformation otherwise known as the Restricted Theory of Relativity has, however, found a place in many a treatise on Electrical theories on account of its simple mathematics (but far less simple ideas) and its immediate application to many classical problems in Optics and Electricity The General Theory which can only be explained with the help of the technical mathematics of Tensor Calculus is not included in any treatise on Mechanics and has remained a subject to be studied only by those who are specially interested Physicists, specially those interested in the experimental side in the English speaking countries, have generally shown very little interest in Relativity, particularly in the General Theory It is therefore refreshing to see a Physicist studying Relativity and we can expect from him physical insight into the intricate mathematical formulae which the Relativitist must use as language for the expression of his ideas

Tolman in his book discusses three different interdependent topics, Relativity, Thermodynamics and Cosmology Though written by a Physicist the book is intensely mathematical But no opportunity has been missed to stress those points where contact has been established with experience At the same time experimental details have been studiously avoided, though it must be admitted that in a comprehensive book like the present one, some more accounts of experiments and observations would have been welcome In presenting the subject the physical nature of the assumptions involved has been placed to the forefront and mathematical analysis has been used to serve physical purpose A notable case in point is the attempt at deducing relations among observable and measurable quantities in the phenomenon of the recession of nebulae in Cosmology, which forms an immensely interesting chapter in this part of the book The

elaborate account of the modern astrophysical methods in chapter X, Part IV, will be found very valuable as these are not easily accessible to the ordinary reader The portion on thermodynamics is entirely the author's contribution to the subject The relativistic analogues of the two laws of Thermodynamics have been proposed by the author and applied to specific cases The author finds that a Schwarzschild sphere of thermodynamic fluid in thermal equilibrium will not have uniform temperature but the temperature would increase towards the centre, so that heat energy (which must have inertia) would be unable to fall from higher to lower potential due to the opposed temperature gradient Other applications of his new thermodynamical equations have been made in the chapters on Cosmology The author has pressed two interesting points of Relativistic Thermodynamics, namely, that relativistic reversible thermodynamic processes with finite velocity are possible and an irreversible process need not culminate in an entropy maximum, and has devoted considerable space to the discussion of these topics. The point of the author in all this development is that Relativistic thermodynamics rather than classical Thermodynamics is likely to give us more success in the grand scale survey of the Universe

The third part of the book dealing with Relativistic Cosmological speculations is very welcome, as such a complete treatment of the subject, specially by one who has worked in it, has not as yet appeared in the literature on Relativity. The cases of the Einstein Universe and the de Sitter Universe have first been fully discussed Then follows the deduction of the nonstatic line element, Lemaitre's Cosmology and complete discussion of the red shift and of the various cosmological models and finally some numerical speculation The treatment in almost all parts of the book is the author's own, which testifies to the thorough acquaintance of the author with the subject even to the minutest details

To the Relativitist the perusal of the book is a real pleasure The introductory portions of many chapters are charming and the author's ways are so convincing that very few readers will be able to complain of any haziness after going through a chapter. It can certainly be said that the literature on Relativity has been considerably enriched by the appearance of Tolman's Relativity, Thermodynamics and Cosmology

N. R. S.

On a new method of calculation of Stark Effect of all orders applicable to the Balmer Lines of Hydrogen

BY

K Basu

(*Communicated by Prof N. R Sen*)

1. *Introductory*

The present paper contains a full account of a new method developed by me for calculation of Stark effect of all orders from consideration of perturbation of a free hydrogen-like atom by external electric force It was pointed out by WENTZEL[1] that the measurements of TAKAMINE and KOKUBU[2] and of KIUTI[3] on the centre of σ-component of H_γ indicate a shift about 20 per cent larger than that given by the old quantum theory according to EPSTEIN[4] and MOSHARRAFA,[5] so far as the 2nd order Stark effect is concerned, but these measurements were found to be in good agreement with the theory as developed by a number of workers on the basis of the Schrodinger mechanics, *viz*, SCHRODINGER,[6] EPSTEIN,[7] WALLER,[8] FOCK,[9] SOMMERFELD,[10] WENTZEL[1] and BASU[11] One important point to be noticed regarding the 2nd order effect derived by these authors is this that their formula, identical in every respect, is confirmed by very accurate experiments by RAUSCH and GEBAUER,[12] and this slightly deviates from the old formula of Epstein and Mosharrafa.

The result arrived at in this paper, so far as the 3rd order term is concerned, exactly tallies with that by ISHIDA and HIYAMA[13] and later workers,[14] mentioned to have been worked out by DOI[15] on the basis of the new wave mechanics The expression of Doi for the 3rd order effect, utilised by Ishida and Hiyama is only approximately true for several π- and σ- components of H_γ and H_δ lines Gebauer and Rausch (l c.) have drawn a calculated and observed Stark effect

shift graph—entitled $\Delta\lambda$- J graph—for the π 18 component of H_γ for the three orders of the field, and pointed out that the deviation of the theoretical from the observed values of $\Delta\lambda$ in the cubic effect is very small for fields ranging from 0, 6 to 1 Million Volt/cm, so that a calculation of the 4th order would be very desirable From calculations of the 4th order term value of energy I have drawn a deviation graph of the same line showing all the four effects, which has upset Gebauer and Rausch's prediction that a 4th order calculation would show hopeful coincidence between theory and experiment

This shows that we cannot utilise perturbation method at least in the case of *very high* field in the Stark effect problem for hydrogen, or for that matter, for hydrogenic atoms, inasmuch as the expression for $\Delta\tilde{\nu}$, *viz*,

$$\Delta\tilde{\nu}=aJ+bJ^2+cJ^3+bJ^4\ldots\ \ldots. \qquad \ldots \quad (1)$$

ceases to be convergent at a certain point for one Million Volt/cm and secondly, it remains valid when the total quantum number is not very large. We shall see that Schroedinger's perturbation theory has these limitations in the present problem, which could not be judged *a priori*

In a more recent paper by LANCZOS[16] an attempt has been made to solve the problem of the Stark effect in hydrogenic atoms for high field by taking an asymptotic solution of the Bessel-type of the wave equation in polar co-ordinates established by Schroedinger instead of the usual parabolic co-ordinates According to this author the usual perturbation method is unsuitable for the problem for some qualitative reasons In the present paper, a few results of which were previously published in abstract,[17] we shall try to substantiate this, quantitatively, from our own standpoint, without having recourse to polar co-ordinates but retaining the parabolic co-ordinates of Epstein[7] and other old workers in this line

2. *A Modified Theory of Perturbation*

The wave equations for the unperturbed and perturbed problems are :—

$$\nabla^2\psi(x_i)+C[E_i-V_0(x_i)]\psi(x_i)=0 \qquad \ldots \quad (2)$$

$$\nabla^2\psi(x_i)+C[E_i-V_0(x_i)-\lambda V_1(x_i)]\psi(x_i)=0 \quad \ldots \quad (3)$$

where C stands for $8\pi^2\mu/h^2$.

Let the characteristic value and the characteristic function of (2) be $E_i = E_i(n)$, and $\psi(x_i) = \psi_0(x_i, n)$ respectively, where, generally, $\psi_0(x_i, n)$ is a polynom of the nth degree in x_i. Assume

$$\psi(x_i) = \sum_{\tau=0}^{\infty} a_\tau \psi_0(x_i, \tau)$$

as a solution of (3), where the series is convergent Since the polynom is well-studied we can express $V_1(x_i)\psi(x_i)$ in a finite series of $\psi_0(x_i, \tau)$ Thus

$$V_1(x_i)\psi(x_i) = \sum_{\tau=0}^{\infty} a_\tau V_1(x_i)\psi_0(x_i, \tau) = \sum_{\tau=0}^{\infty} a_\tau [f_\tau^{-j}\psi_0(x_i, \tau-j) +$$

$$+ f_\tau^{0}\psi_0(x_i, \tau) + \dots + f_\tau^{j}\psi_0(x_i, \tau+j)]$$

where the co-efficients $f_\tau^k (k = -j, \dots, 0, \dots +j)$ are all known

Substituting in (3) we get ·

$$\nabla^2 \sum_{\tau=0}^{\infty} a_\tau \psi_0(x_i, \tau) + C[E_i - V_0(x_i)] \sum_{\tau=0}^{\infty} a_\tau \psi_0(x_i, \tau)$$

$$= C\lambda \sum_{\tau=0}^{\infty} a_\tau [f_\tau^{-j}\psi_0(x_i, \tau-j) + \text{ etc }] \qquad (3')$$

Now since $\psi_0(x_i, \tau)$ satisfies (2) we get

$$\nabla^2\psi_0(x_i, \tau) + C[E_i(\tau) - V_0(x_i)]\psi_0(x_i, \tau) = 0$$

$$(\tau = 0, 1, \dots) \qquad (2')$$

Subtracting (2′) from (3′) we derive an identity

$$\sum_{\tau=0}^{\infty} [E_i - E_i(\tau)] a_\tau \psi_0(x_i, \tau) \equiv \lambda \sum_{\tau=0}^{\infty} a_\tau [f_\tau^{-j}\psi_0(x_i, \tau-j) + \dots]$$

Equate co-efficients of $\psi_0(x_i, \tau)$ to zero ·

$$[E_i - E_i(\tau) - \lambda f_\tau^0] a_\tau - \lambda a_{\tau+j} f_{\tau+j}^{-j} - \dots - \lambda a_{\tau-j} f_{\tau-j}^{j} = 0,$$

$$(\tau = 0, 1, 2, \dots) \qquad (5)$$

We get then an infinite set of homogeneous linear equations in the co-efficients a_τ's. Since the co-efficients are assumed to be not vanishing we get the following determinant Δ to be zero

$$
\Delta \equiv \left| \begin{array}{cccccc}
 & \ddots & & & & \\
, & \theta-\lambda f^{0}_{n-2}, & -\lambda f^{-1}_{n-1}, & -\lambda f^{-2}_{n}, & -\lambda f^{-3}_{n+1}, & -\lambda f^{-4}_{n+2}, \\
\cdot\,, & -\lambda f^{1}_{n-2}, & \theta-\lambda f^{0}_{n-1}, & -\lambda f^{-1}_{n}, & -\lambda f^{-2}_{n+1}, & -\lambda f^{-3}_{n+2}, \quad . \\
, & -\lambda f^{2}_{n-2}, & -\lambda f^{1}_{n-1}, & \boxed{\theta-\lambda f^{0}_{n}}, & -\lambda f^{-1}_{n+1}, & -\lambda f^{-2}_{n+2}, \\
\ldots, & -\lambda^{3}_{n-2}, & -\lambda f^{2}_{n-1}, & -\lambda f^{1}_{n}, & \theta-\lambda f^{0}_{n+1}, & -\lambda f^{-1}_{n+2}, \\
, & -\lambda f^{4}_{n-2}, & -\lambda f^{3}_{n-1}, & -\lambda f^{2}_{n}, & -\lambda f^{1}_{n+1}, & \theta-\lambda f^{0}_{n+2}, \quad .
\end{array} \right| = 0
$$

In the above we have written $\tau = n$, a particular integer, and $\theta = E_{\cdot} - E_{\cdot}(n)$ where the *eigenvalue* θ is to be determined. It is clear that, when $\lambda = 0$, $\theta = 0$, and when $\lambda \neq 0$, $\theta = \sum\limits_{r=1}^{\infty} \lambda^r \theta_{r,n}$, where $\theta_{r,n}$ etc., are to be found out in terms of the known f's by expanding the determinant. The crux of the problem is to solve this determinantal equation of infinite order for the proper value θ. Generally a root of θ in the vicinity of $\theta = 0$ will serve our purpose.

An exhaustive study of infinite determinants was made by von Koch. We shall follow his procedure for expansion of such determinants. Koch has discussed the general case when n runs from $-\infty$ to $+\infty$, but in our case n runs from 0 to $+\infty$. Each row of our determinant contains $(2\eta+1)$ elements. If we divide each row by the corresponding diagonal element of that row all the elements of the leading diagonal become unity, and the determinant is then of the form:

$$
\Delta \equiv \left| \begin{array}{cccccc}
 & & \ddots & & & \\
, & A_{i-1,i-2}, & 1\,, & A_{i-1,i}, & A_{i-1,i+1}, & A_{i-1,i+2}, \\
, & A_{i,i-2}, & A_{i,i-1}, & 1\,, & A_{i,i+1}, & A_{i,i+2}, \quad . \\
\cdot\,, & A_{i+1,i-2}, & A_{i+1,i-1}, & A_{i+1,i}, & 1\,, & A_{i+1,i+2}, \quad . \\
 & \ldots & \ldots & \ldots & &
\end{array} \right| = 0
$$

Now this will be of the normal type if the doubly infinite series $\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}|A_{l,k}|$ converges Let us expand this determinant by the scheme noted below:

$$\Delta\equiv 1+\Sigma\begin{vmatrix}0, & (ij)\\ (ji), & 0\end{vmatrix}+\Sigma\begin{vmatrix}0, & (ij), & (ik)\\ (ji), & 0, & (jk)\\ (ki), & (kj), & 0\end{vmatrix}+\Sigma\begin{vmatrix}0, & (ij), & (ik), & (il)\\ (ji), & 0, & (jk), & (jl)\\ (ki), & (kj), & 0, & (kl)\\ (li), & (lj), & (lk), & 0\end{vmatrix}+$$

where we have written (ij) for $A_{i,j}$, and i, j, k, l, are such that $i<j<k<l$. It will be clear that the 2nd order determinant contains λ^2, the 3rd order one contains λ^3, and so on, so that we can find out perturbations up to any desired order

3 *Formation of Δ in Stark-effect*

We shall not describe here the details as to how the Schrodinger wave equation for Stark-effect problem is formed in parabolic co-ordinates but shall give an outline The reader is referred to Sommerfeld's treatise on 'Wave Mechanics' (Chap II, § 2A)

The usual Schrödinger equation for this case is

$$\nabla^2\psi+\frac{8\pi^2m_0}{h^2}\left(E+\frac{Ze^2}{r}\right)\psi=\lambda x\psi \tag{6}$$

where $\lambda=8\pi^2m_0eJ/h^2$, J=external field in the direction of the x-axis Using the usual parabolic co-ordinates (ξ, η) defined by

$$x=\tfrac{1}{2}(\xi^2-\eta^2),\ y=\xi\eta\cos\phi,\ z=\xi\eta\sin\phi,$$

and assuming

$$\psi=f_1(\xi)f_2(\eta)e^{\pm im\phi}$$

the combined differential equation for f_1 and f_2 is:

$$\frac{d^2f}{dr^2}+\frac{1}{r}\frac{df}{dr}+\left(A+\frac{2B}{r}+\frac{C}{r^2}\right)f=\lambda rf. \tag{7}$$

wherein we have introduced the abbreviations

$$r=\begin{cases}\xi^2\\ \eta^2\end{cases}, \qquad f=\begin{cases}f_1\\ f_2\end{cases}, \qquad \lambda=\pm\pi^2 m_0 eJ/h^2,$$

$$A=2\pi^2 m_0 E/h^2,\ B=\frac{\pi^2 m_0}{h^2}\,(Ze^2\mp\beta),\quad C=-\,\frac{m^2}{4}$$

(β=separation const)

Using the transformation

$$f=e^{-\frac{\rho}{2}}\,\rho^{\frac{m}{2}}.w(\rho),\ (\rho=2\sqrt{-A}\ r)$$

we ultimately arrive at the differential equation :

$$\rho w''(\rho)+(m+1-\rho)w'(\rho)+n_1 w=\lambda'\rho^2 w \qquad \dots \quad (8)$$

wherein for brevity

$$n_1=\frac{B}{\sqrt{-A}}-\tfrac{1}{2}(m+1),\quad \lambda'=\frac{\lambda}{(2\sqrt{-A})^3}\ .$$

The solution of equation (8), when λ' is absent, is well-known to be a derivative of the Laguerre polynom,[2,5] which we shall denote by the symbol $L^m_{m+n_1}$, and call this an "associated Laguerre polynom of degree n_1 and derivation m" For the solution of the complete equation (8) we assume a series involving these polynoms. Thus

$$w(\rho)=\sum_{r=0}^{\infty} a_{m+r}L^m_{m+r}(\rho) \qquad \dots \quad (9)$$

Substituting this in (8) and remembering that

$$\rho L''^m_{m+r}(\rho)+(m+1-\rho)L'^m_{m+r}(\rho)+rL^m_{m+r}(\rho)=0 \qquad (r=0,\ 1,2\dots)$$

we derive :

$$\sum_{r=0}^{\infty} a_{m+r}(n_1-r)L^m_{m+r}(\rho)=\lambda'\sum_{r=0}^{\infty} a_{m+r}\rho^2 L^m_{m+r}(\rho) \qquad \dots \quad (10)$$

the right hand side of this admits of expansion in terms of associated Laguerre polynoms as can be seen from a recurring formula established below :

The Laguerre-polynom $L_n(x)$ as defined by Courant-Hilbert is developed from the following generating function (for $t<1$)

$$\frac{e^{\frac{-xt}{1-t}}}{1-t}=\sum_{n=0}^{\infty}\frac{L_n(x)t^n}{n!}$$

From this it is easily seen that the associated polynoms possess a generating function defined by

$$\frac{e^{\frac{-xt}{1-t}}}{(1-t)^{m+1}}=(-)^m\sum_{n=0}^{\infty}\frac{L^m_{n+m}(x)t^n}{(n+m)!}\equiv\psi(x,t).$$

This is in fact clear by differentiating m times the corresponding Courant-Hilbert identity for L_{n+m} with regard to t. Taking logarithms and differentiating with respect to t we obtain :

$$[(m+1)(1-t)-x]\psi=(1-t)^2\psi',$$

where ψ' denotes the first derivative of ψ with regard to t. Substituting for ψ and ψ' their corresponding values in series given above, and equating co-efficients of t^n from both sides we get the sequence relation

$$\frac{n+1}{n+m+1}L^m_{n+m+1}(x)-(2n+m+1-x)L^m_{n+m}(x)+(n+m)^2L^m_{n+m-1}(x)=0.$$

By repeated application of this we find that $x^2L^m_{n+m}(x)$ is expressible in terms of five polynoms with constant co-efficients, of degrees $n-2$, $n-1$, n, $n+1$, $n+2$. Thus

$$x^2\ L^m_{n+m}(x)=\frac{(n+1)(n+2)}{(n+m+1)(n+m+2)}L^m_{n+m+2}(x)$$

$$-\frac{2(n+1)(2n+m+2)}{n+m+1}L^m_{n+m+1}(x)+(6n^2+6nm+6n+m^2$$

$$+3m+2)L^m_{n+m}(x)-2(n+m)^2.\ (2n+m)L^m_{n+m-1}(x)$$

$$+(n+m)^2(n+m-1)^2L^m_{n+m-2}(x).$$

Applying this result to the right hand number of (10) and equating co-efficients of say $L^m_{r+m}(\rho)$ from both sides we obtain .

$$\beta_{r-2}a_{m+r-2}+\gamma_{r-1}a_{m+r-1}+(\theta+a_r)a_{m+r}+\delta_{r+1}a_{m+r+1}+\epsilon_{r+2}a_{m+r+2}=0 \quad \cdots \quad (11)$$

where

$$\theta=-n_r,\ a_r=\eta+\lambda' A_r,\ \beta_r=\lambda'\beta'_r,\ \gamma_r=\lambda'\gamma'_r,\ \delta_r=\lambda'\delta'_r,\ \epsilon_r=\lambda'\epsilon'_r,$$

$$\beta'_r=\frac{(r+1)(r+2)}{(r+m+1)(r+m+2)},\ \gamma'_r=-\frac{2(r+1)(2r+m+2)}{r+m+1},$$

$$a'_r\equiv A_r=(6r^2+6rm+6r+m^2+3m+2),$$

$$\delta'_r=-2(r+m)^2(2r+m),\ \epsilon'_r=(r+m)^2(r+m-1)^2$$

We have thus an infinite set of homogeneous equations for the unknown co-efficients $a_m, a_{m+1}, a_{m+2},$ The condition that such sets of equations have a solution, which do not vanish identically, is that the determinant of the co-efficients, $\Delta[\theta]$, should vanish The *eigenwert* θ is to be found out from the following determinantal equation—

$$\Delta[\theta]=\begin{vmatrix} & & & \cdot & & & & \\ ,\gamma_{r-3}, & \theta+a_{r-2}, & \delta_{r-1}, & \epsilon_r, & 0, & 0, & 0, & \cdots \\ ,\beta_{r-3}, & \gamma_{r-2}, & \theta+a_{r-1}, & \delta_r, & \epsilon_{r+1}, & 0, & 0, & \\ ,\ 0, & \beta_{r-2}, & \gamma_{r-1}, & \boxed{\theta+a_r}, & \delta_{r+1}, & \epsilon_{r+2}, & 0, & \\ ,\ 0, & 0, & \beta_{r-1}, & \gamma_r, & \theta+a_{r+1}, & \delta_{r+2}, & \epsilon_{r+3}, & \cdot \\ ,\ 0, & 0, & 0, & \beta_r, & \gamma_{r+1}, & \theta+a_{r+2}, & \delta_{r+3}, & \cdot \\ & & & \cdot & & & \cdot & \end{vmatrix}0,\ r\{^0_\infty\}$$

Since we are concerned with that root of θ which lies in the vicinity of $\theta=-a_r$ we have written the determinant in this form placing the element $\theta+a_r$ exactly at the centre, and bordered The root we are going to find out is developed from the root $-a_r$.

4. *Expansion of the Determinant*

In order to facilitate calculations for high orders, say the 3rd, and 4th orders we shall frame a supplementary determinant Δ' out of Δ. This is obtained by writing dotted non-diagonal elements

instead of non-dotted elements This means λ' is absent from all non-diagonal elements The following notations are utilised for writing this determinant

$$\beta'_r \equiv [r+2, r] = (20)\,,\ \beta'_{r-2} \equiv [r, r-2] = (0\bar{2})\,,$$

$$\beta'_{r-1} \equiv [r+1, r-1] = (1\bar{1})\,,\ \beta'_{r+2} = (42)\,, \text{ etc}$$

$$\gamma'_r \equiv [r+1, r] = (10)\,,\ \gamma'_{r-1} \equiv [r, r-1] = (0\bar{1})\,,$$

$$\gamma'_{r+1} \equiv [r+2, r+1] = (21)\,, \text{ etc}$$

$$\delta'_{r+1} \equiv (r, r+1] = (01)\,,\ \delta'_r \equiv [r-1, r] = (\bar{1}0)\,,$$

$$\delta'_{r-1} \equiv [r-2, r-1] = (\bar{2}\bar{1})\,, \text{ etc}$$

$$\epsilon'_{r+2} \equiv [r, r+2] = (02)\,,\ \epsilon'_r \equiv [r-2, r] = (\bar{2}0)\,,$$

$$\epsilon'_{r+1} \equiv [r-1, r+1] = (\bar{1}1)\,, \text{ etc}$$

we have then

$$\begin{vmatrix} & \cdots & \cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & & & & & \\ \cdots, & (\overline{44}), & (\overline{43}), & (\overline{42}), & 0, & 0, & 0, & 0, & 0, & 0, & \cdots \\ , & (\overline{34}), & (\overline{33}), & (\overline{32}), & (\overline{31}), & 0, & 0, & 0, & 0, & 0, & \cdots \\ \cdots, & (\overline{24}), & (\overline{23}), & (\overline{22}), & (\overline{21}), & (\bar{2}0), & 0, & 0, & 0, & 0, & \cdot \\ \cdot, & 0, & (\overline{13}), & (\overline{12}), & (\overline{11}), & (\bar{1}0), & (\bar{1}1), & 0, & 0, & 0, & \cdots \\ , & 0, & 0, & (0\bar{2}), & (0\bar{1}), & (00), & (01), & (02), & 0, & 0, & \cdots \\ \cdots, & 0, & 0, & 0, & (1\bar{1}), & (10), & (11), & (12), & (13), & 0, & \cdots \\ , & 0, & 0, & 0, & 0, & (20), & (21), & (22), & (23), & (24), & \cdots \\ \cdots, & 0, & 0, & 0, & 0, & 0, & (31), & (32), & (33), & (34), & \\ \cdot, & 0, & 0, & 0, & 0, & 0, & 0, & (42), & (43), & (44), & \cdot \\ & \cdots & \cdots & & & & & & \cdot & & \end{vmatrix} = 0$$

Make the elements of the leading diagonal in both the determinants unity, and expand them according to the scheme given in Sec. 2 This gives ;

$$\Delta[\theta]=1+\frac{\lambda'^2\Sigma S_{ij}}{(\theta+a_i)\ (\theta+a_j)}+\frac{\lambda'^3\Sigma S_{ijk}}{(\theta+a_i)\ (\theta+a_j)\ (\theta+a_k)}$$

$$+\frac{\lambda'^4\Sigma S_{ijkl}}{(\theta+a_i)(\theta+a_j)(\theta+a_k)\ (\theta+a_l)}+ \qquad (12)$$

where

$$S_{ij}=\begin{vmatrix} 0 & ,(ij) \\ (ji), & 0 \end{vmatrix}\ ; \quad S_{ijk}=\begin{vmatrix} 0 & ,(ij)\ (ik) \\ (ji)\ , & 0\ ,(jk) \\ (ki)\ ,(kj), & 0 \end{vmatrix};$$

$$S_{ijkl}=\begin{vmatrix} 0 & ,(ij), & (ik), & (il) \\ (ji)\ , & 0 & ,(jk), & (jl) \\ (ki)\ , & (kj), & 0 & ,(kl) \\ (li)\ , & (lj), & (lk)\ , & 0 \end{vmatrix}$$

Equa. (12) may be written more clearly as

$$\Big[\theta+a_r\Big]\Big[1+\lambda'^2\Sigma\frac{S_{ij}}{(\theta+a_i)\ (\theta+a_j)}+\lambda'^3\Sigma\frac{S_{ijk}}{(\theta+a_i)\ (\theta+a_k)(\theta+a_j)}$$

$$+\lambda'^4\Sigma\ \frac{S_{ijkl}}{(\theta+a_i)(\theta+a_j)(\theta+a_k)\ (\theta+a_l)}\Big]$$

$$+\lambda'^2\sum_j\frac{S_{rj}}{\theta+a_j}+\lambda'^3\sum_{j,k}\frac{S_{rjk}}{(\theta+a_j)(\theta+a_k)}+\lambda'^4\sum_{j,k,l}\frac{S_{rjkl}}{(\theta+a_j)(\theta+a_k)(\theta+a_l)}=0,$$

$$or,\qquad \Big[\theta+a_r\Big]+\Big[\lambda'^2\sum_j\ \frac{S_{rj}}{\theta+a_j}+\lambda'^3\sum_{j,k}\frac{S_{rjk}}{(\theta+a_j)(\theta+a_k)}$$

$$+\lambda'^4\sum_{j,k,l}\frac{S_{rjkl}}{(\theta+a_j)\ (\theta+a_k)\ (\theta+a_l)}\Big]\Big[1-\lambda'^2\sum_{\iota,\nu}\frac{S_{\iota\nu}}{(\theta+a_\iota)(\theta+a_\nu)}\Big]=0\ . \quad (13)$$

keeping up to the 4th order in λ',

$$or,\qquad \theta+r+\lambda' A_r+\lambda'^2\sum_j\frac{S_{rj}}{\theta+a_j}+\lambda'^3\sum_{j,k}\frac{S_{rjk}}{(\theta+a_j)(\theta+a_k)}$$

$$+\lambda'^4 \sum_{j,k,l} \frac{S_{rjkl}}{(\theta+a_j)(\theta+a_k)(\theta+a_l)} - \lambda'^4 \sum_j \frac{S_{rj}}{\theta+a_j} \sum_{\iota,\nu} \frac{S_{\iota\nu}}{(\theta+a_\iota)(\theta+a_\nu)} = 0 \quad (14)$$

The following are the proper values for successive approximations —

$$\theta_0 = -r,$$

$$\theta_1 = -r - \lambda' A_r,$$

$$\theta_2 = -r - \lambda' A_r - \lambda'^2 \sum_j \frac{S_{rj}}{\theta+a_j} = -r - \lambda' A_r - \lambda'^2 \sum_j \frac{S_{rj}}{(j-r)},$$

$$\theta_3 = -r - \lambda' A_r - \lambda'^2 \sum_j \frac{S_{rj}}{[(j-r)+\lambda'(A_j - A_r)]} - \lambda'^3 \sum_{j,k} \frac{S_{rjk}}{(j-r)(k-r)},$$

$$= -r - \lambda' A_r - \lambda'^2 \sum_j \frac{S_{rj}}{(j-r)} \left[1 - \lambda' \frac{(A_j - A_r)}{(j-r)}\right] - \lambda'^3 \sum_{j,k} \frac{S_{rjk}}{(j-r)(k-r)},$$

$$= -r - \lambda' A_r - \lambda'^2 \sum_j \frac{S_{rj}}{(j-r)} + \lambda'^3 \left[\sum_j \frac{S_{rj}(A_j - A_r)}{(j-r)^2} - \sum_{j,k} \frac{S_{rjk}}{(j-r)(k-r)}\right],$$

$$\theta_4 = -r - \lambda' A_r - \lambda'^2 \sum_j \frac{S_{rj}}{\left[(j-r)+\lambda'(A_j - A_r) - \lambda'^2 \sum_j \frac{S_{rj}}{(j-r)}\right]}$$

$$-\lambda'^3 \sum_{j,k} \frac{S_{rjk}}{[(j-r)+\lambda'(A_j - A_r)][(k-r)+\lambda'(A_k - A_r)]}$$

$$-\lambda'^4 \sum_{j,k,l} \frac{S_{rjkl}}{(j-r)(k-r)(l-r)} + \lambda'^4 \sum_j \frac{S_{rj}}{(j-r)} \sum_{\iota,\nu} \frac{S_{\iota\nu}}{(\iota-r)(\nu-r)},$$

$$= -r - \lambda' A_r - \lambda'^2 \sum_j \frac{S_{rj}}{(j-r)} \left[1 + \lambda' \frac{A_j - A_r}{(j-r)} - \lambda'^2 \sum_j \frac{S_{rj}}{(j-r)^2}\right]^{-1}$$

$$-\lambda'^3 \sum_{j,k} \frac{S_{rjk}}{(j-r)(k-r)} \left[1 + \lambda' \frac{A_j - A_r}{(j-r)}\right]^{-1} \left[1 + \lambda' \frac{A_k - A_r}{(k-r)}\right]^{-1}$$

$$-\lambda'^4 \sum_{j,k,l} \frac{S_{rjkl}}{(j-r)(k-r)(l-r)} + \lambda'^4 \sum_j \frac{S_{rj}}{(j-r)} \sum_{\iota,\nu} \frac{S_{\iota\nu}}{(\iota-r)(\nu-r)};$$

$$=-r-\lambda' A_r-\lambda'^2\sum_j \frac{S_{rj}}{(j-r)}\left[1-\lambda'\frac{A_j-A_r}{(j-r)}+\lambda'^2\frac{1}{(j-r)}\sum_j\frac{S_{rj}}{(j-r)}\right.$$

$$\left.+\lambda'^2\frac{(A_j-A_r)^2}{(j-r)^2}\right]-\lambda'^3\sum_{j,k}\frac{S_{rjk}}{(j-r)(k-r)}\left[1-\lambda'\left(\frac{A_j-A_r}{j-r}+\frac{A_k-A_r}{k-r}\right)\right]$$

$$-\lambda'^4\sum_{j,k,l}\frac{S_{rjkl}}{(j-r)(k-r)(l-r)}+\lambda'^4\sum_j\frac{S_{rj}}{(j-r)}\sum_{\iota,\nu}\frac{S_{\iota\nu}}{(\iota-r)(\nu-r)};$$

$$=-r-\lambda' A_r-\lambda'^2\sum_j\frac{S_{rj}}{(j-r)}+\lambda'^3\left[\sum_j\frac{S_{rj}(A_j-A_r)}{(j-r)^2}-\sum_{j,k}\frac{S_{rjk}}{(j-r)(k-r)}\right]$$

$$+\lambda'^4\left[-\sum_j\frac{S_{rj}(A_j-A_r)^2}{(j-r)^3}+\sum_j\frac{S_{rj}}{(j-r)}\sum_{\iota,\nu}\frac{S_{\iota\nu}}{(\iota-r)(\nu-r)}\right.$$

$$-\sum_j\frac{S_{rj}}{(j-r)^2}\sum_\lambda\frac{S_{r\lambda}}{(\lambda-r)}+\sum_{j,k}\frac{S_{rjk}(A_j-A_r)}{(j-r)^2(k-r)}$$

$$\left.+\sum_{j,k}\frac{S_{rjk}(A_k-A_r)}{(j-r)(k-r)^2}-\sum_{j,k,l}\frac{S_{rjkl}}{(j-r)(k-r)(l-r)}\right]$$

We give now the equivalents of the determinants S's in expanded form, which will be useful afterwards

$$S_{rj}=-(rj)\,(jr),\quad S_{rjk}=(rj)\,(jk)\,(kr)+(rk)\,(kj)\,(jr),$$

$$S_{rjkl}=[(rj)\,(jr)\,(kl)\,(lk)+(rl)\,(lr)\,(jk)\,(kj)+(rk)\,(kr)\,(jl)\,(lj)]$$

$$-[(rj)\,(jk)\,(kl)\,(lr)+(rj)\,(jl)\,(lk)\,(kr)+(rk)\,(kl)\,(lj)\,(jr)$$

$$+(rk)\,(kj)\,(jl)\,(lr)+(rl)\,(lk)\,(kj)\,(jr)+(rl)\,(lj)\,(jk)\,(kj)].$$

5. *Calculation of Energy up to Second Order*

With the help of results obtained in the preceding Sec for the proper values of different orders we easily write down now the proper value of the 2nd order

$$\theta_2=-r-\lambda' A_r+\lambda'^2\sum_j\frac{(rj)\,(jr)}{(j-r)},$$

$$=-r-\lambda' A_r+\lambda'^2\left\{\frac{(01)\,(10)}{1}+\frac{(02)\,(20)}{2}+\frac{(0\bar{1})\,(\bar{1}0)}{-1}+\frac{(0\bar{2})\,(\bar{2}0)}{-2}\right\},$$

$$=-r-\lambda' A_r+\lambda'^2\{\delta_{r+1}\gamma'_r+\tfrac{1}{2}\epsilon'_{r+2}\beta'_r-\gamma'_{r-1}\delta'_r-\tfrac{1}{2}\beta'_{r-2}\epsilon'_r\},$$

$$=-r-\lambda' A_r+\lambda'^2[4\{(r+m+1)(2r+m+2)^2(r+1)-r(r+m)(2r+m)^2\}$$

$$+\tfrac{1}{2}\{(r+1)(r+2)(r+m+1)(r+m+2)-r(r-1)(r+m)(r+m-1)\}] \quad (15)$$

It is interesting to note that the four terms within the above parentheses are identical with the corresponding terms of Sommerfeld (*l c* p 191) obtained by Schrodinger's method of Storungs-calculation, of course, there will be a change of sign on account of the fact that our θ corresponds to $-n$,

Adapting this result (15) to the constants of the two differential equations in f_1 and f_2, and assigning proper signs to λ' we write

$$\frac{\pi^2 m_0}{h^2\sqrt{-A}}(Ze^2-\beta)-\tfrac{1}{2}(m+1)=r_1+\lambda'(6r_1^2+6r_1m+6r_1+m^2+3m+2)$$

$$-\lambda'^2[4\{(r_1+1)(r_1+m+1)(2r_1+m+2)^2-r_1(r_1+m)(2r_1+m)^2\}$$

$$+\tfrac{1}{2}\{(r_1+1)(r_1+2)(r_1+m+1)(r_1+m+2)-r_1(r_1-1)$$

$$(r_1+m-1)(r_1+m)\}] \quad (16)$$

$$\frac{\pi^2 m_0}{h^2\sqrt{-A}}(Ze^2+\beta)-\tfrac{1}{2}(m+1)=r_2-\lambda'(6r_2^2+6r_2m+6r_2+m^2+3m+2)$$

$$-\lambda'^2[4\{(r_2+m+1)(r_2+1)(2r_2+m+2)^2-r_2(r_2+m)(2r_2+m)^2\}$$

$$+\tfrac{1}{2}\{(r_2+1)(r_2+2)(r_2+m+1)(r_2+m+2)-r_2(r_2-1)(r_2+m-1)$$

$$(r_2+m)\}] \quad . (17)$$

Adding up (16) and (17) and making algebraical calculations we find r_1+r_2+m+1 is a factor We derive finally

$$\frac{2\pi^2 m_0 Ze^2}{h^2\sqrt{-A}}=n\Big[1+\frac{6\lambda}{(2\sqrt{-A})^3}(r_1-r_2)-\frac{2\lambda^2}{(2\sqrt{-A})^6}$$

$$\{4m^2+17m(r_1+r_2+1)+34(r_1^2+r_2^2-r_1r_2)+17(r_1+r_2)+18\}\Big], \quad . (18)$$

wherein we have put $n=r_1+r_2+m+1$

The result tallies with that given in equa (33) of Sommerfeld's book (l c p 191) if we read n_1, n_2 for r_1, r_2 Now putting in the values of λ in terms of the field-strength J we are led to the following energy value from a simple calculation

$$E=-\frac{RhZ^2}{n^2}+\frac{3h^2J}{8\pi^2 m_0 eZ}\,n(n_1-n_2)-\frac{h^6J^2}{16(2\pi e)^6 m_0^3 Z^4}\,n^4$$

$$[17n^2-3(n_1-n_2)^2-9m^2+19] \quad . \qquad (19)$$

6 *Calculation of Energy up to Third Order*

With the help of the expression for θ_3 as given in Sec 4 we write down :

$$\theta_3=-r-\lambda'A_r-\lambda'^2\sum_j\frac{S_{rj}}{j-r}+\lambda'^3\left[\sum_j\frac{S_{rj}(A_j-A_r)}{(j-r)^2}-\sum_{j,k}\frac{S_{rjk}}{(j-r)(k-r)}\right]$$

$$=\theta_2+\lambda'_3\left[-\sum_j\frac{(rj)(jr)(A_j-A_r)}{(j-r)^2}-\sum_{j,k}\frac{(rj)(jk)(kr)+(rk)(kj)(jr)}{(j-r)(k-r)}\right]$$

$$=\theta_2+\lambda'^3\left[-\frac{(01)(10)(A_{r+1}-A_r)}{1^2}-\frac{(02)(20)(A_{r+2}-A_r)}{2^2}\right.$$

$$-\frac{(0\bar{1})(\bar{1}0)(A_{r-1}-A_r)}{1^2}-\frac{(0\bar{2})(\bar{2}0)(A_{r-2}-A_r)}{2^2}$$

$$-\frac{(01)(12)(20)+(02)(21)(10)}{1.2}-\frac{(0\bar{1})(\bar{1}\bar{2})(\bar{2}0)+(0\bar{2})(\bar{2}\bar{1})(\bar{1}0)}{(-1)(-2)}$$

$$\left.-\frac{(01)(1\bar{1})(\bar{1}0)}{1\;-1}-\frac{(0\bar{1})(\bar{1}1)(10)}{-1\;1}\right],$$

$$=\theta_2-\lambda'^3[\delta'_{r+1}\gamma'_r(A_{r+1}-A_r)+\gamma'_{r-1}\delta'_r(A_{r-1}-A_r)$$

$$+\tfrac{1}{4}\epsilon'_{r+2}\beta'_r(A_{r+2}-A_r)+\tfrac{1}{4}\beta'_{r-2}\epsilon'_r(A_{r-2}-A_r)+\tfrac{1}{2}\{(\delta'_{r+1}\delta'_{r+2}\beta'_r$$

$$+\gamma'_r\gamma'_{r+1}\epsilon'_{r+2})+(\gamma'_{r-1}\gamma'_r\epsilon'_r+\delta'_{r-1}\delta'_r\beta'_{r-2})-(\delta'_{r+1}\delta'_r\beta'_{r-1}$$

$$+\gamma'_{r-1}\gamma'_r\epsilon'_{r+1})\}].$$

Substituting the values of $A_{r+1}-A_r$, $A_{r-1}-A_r$, $A_{r+2}-A_r$, A_{r-2} $-A_r$, as given in (11), we write down the coefficient of λ'^3 :

$$3\{(2r+m-1)\ \beta'_{r-2}\epsilon'_r-(2r+m+3)\ \beta'_r\epsilon'_{r+2}\}+6\{(2r+m)\delta'_r\gamma'_{r-1}$$

$$-(2r+m+2)\delta'_{r+1}\gamma'_r\}+\tfrac{1}{2}(\delta'_{r-1}\delta'_r\beta'_{r-2}+\epsilon'_r\gamma'_{r-1}\gamma'_{r-2}+\beta'_r\delta'_{r+1}\delta'_{r+2}$$

$$+\epsilon'_{r+2}\gamma'_r\gamma'_{r+1})-(\beta'_{r-1}\delta'_r\delta'_{r+1}+\epsilon'_{r+1}\gamma'_r\gamma'_{r-1}) \qquad . \quad (20)$$

Next, substituting the proper values of β', γ', . as given in (11), we give now the coefficients of $-\lambda'^3$ in abridged form as

$$4\{(\Theta_{r+2}-\Theta_{r+1})-(\Theta_{r+1}-\Theta_r)\}+3(\Phi_{r+2}-\Phi_r)+24(\Psi_{r+1}-\Psi_r), \qquad (21)$$

wherein we have denoted

$$\Theta(r)\equiv r(r-1)(r+m)(r+m-1)(2r+m)(2r+m-2)\ ,$$

$$\Phi(r)\equiv r(r-1)(r+m)(r+m-1)(2r+m-1)\ ;$$

$$\psi(r)\equiv r(r+m)(2r+m)^3.$$

After some labour the expression (21) is reduced to a biquadratic $G(r)$ in r as given below :

$$G(r)\equiv1500r^4+3000(m+1)r^3+(1992m^2+4500m+3168)r^2+(492m^3$$

$$+1992m^2+3168m+1668)r+(9m^4+85m^3+278m^2+374m+172)\ \ldots \quad (22)$$

Remembering $\theta=-n_i$, where $n_i=\dfrac{B}{\sqrt{-A}}-\tfrac{1}{2}(m+1)$, and n_1 has two-fold values corresponding to the two values of B in the $\xi-$ and $\eta-$ equations in parabolic co-ordinates, we write down :

$$\frac{2\pi^2m_0Ze^2}{h^2\sqrt{-A}}=n\left[\ 1+\frac{\lambda}{(2\sqrt{-A})^3}\ g_1(n_1,\ n_2)-\frac{\lambda^2}{(2\sqrt{-A})^6}\ g_2(n_1,n_2)\right.$$

$$\left.+\frac{\lambda^3}{(2\sqrt{-A})^9}\ g_3(n_1,\ n_2)\ \right] \qquad \ldots \quad (23)$$

where $\quad n=n_1+n_2+m+1\,, \qquad g_1=6(n_1-n_2)\,;$

$$g_2=2\{4m^2+17(n_1+n_2+1)m+34(n_1^2+n_2^2-n_1n_2)+17(n_1+n_2)+18\}\,;$$

$$ng_3=G(n_1)-G(n_2)$$

$$=6n(n_1-n_2)[250(n_1^2+n_2^2)+250(m+1)n-168m^2-250m+28]\,,$$

obtained after some calculation

Put $2\sqrt{-A}=K$, and $4\pi^2 m_0 Ze^2/h=c$, and rewrite equation (23) as

$$K=c/n\left[1+\lambda\,\frac{g_1}{K^3}-\lambda^2\,\frac{g_2}{K^6}+\lambda^3\,\frac{g_3}{K^9}\right] \tag{24}$$

Evaluation of K can be carried out by successive approximations Thus, denoting K_0, K_1, K_2, K_3 as the values of K, (*i*) when λ is neglected, (*ii*) when its 1st order is taken into account, (*iii*) when its 2nd order is counted, and (*iv*) when its 3rd order is counted respectively, we find :

$$K_0=c/n\,,$$

$$K_1=c/n\left[1+\lambda\,\frac{g_1}{K_0^3}\right]=c/n\left[1+\lambda g_1.\frac{n^3}{c^3}\right];$$

$$K_2=c/n\left[1+\lambda\,\frac{g_1}{K_1^3}-\lambda^2\,\frac{g_2}{K_0^6}\right]=c/n\left[1+\lambda\,\frac{n^3g_1}{c^3}+\lambda^2\,\frac{n^6}{c^6}(3g_1^2-g_2)\right];$$

$$K_3=c/n\left[1+\lambda\,\frac{g_1}{K_2^3}-\lambda^2\,\frac{g_2}{K_1^6}+\lambda^3\,\frac{g_3}{K_0^9}\right]=c/n\left[1+\lambda\,\frac{n^3}{c^3}\,g_1\right.$$

$$\left.+\lambda^2\,\frac{n^6}{c^6}\,(3g_1^2-g_2)+\lambda^3\,\frac{n^9}{c^9}(g_3-9g_1g_2+12g_1^3)\right].$$

Remembering $A=2\pi^2m_0E/h^2$, we derive :

$$E=-\frac{2\pi^2m_0e^4Z^2}{h^2n^2}\left[1-2\lambda\,\frac{n^3}{c^3}g_1+\lambda^2\,\frac{n^6}{c^6}\,(2g_2-3g_1^2)\right.$$

$$\left.-\lambda^3\,\frac{n^9}{c^9}\,(10g_1^3-12g_1g_2+2g_3)\right] \quad .. \quad (25)$$

wherein λ should be expressed in terms of J.

Since we have calculated the 2nd order effect in sec 5 we are concerned here with the 3rd order term value of energy We write this down in full below :

$$+\frac{12h^{10}J^{3}n^{7}(n_1-n_2)}{2^{17}\pi^{10}.\,m_0^5\,e^{11}\,Z^7}\Big[180(n_1-n_2)^2-12\{4m^2+17m(n_1+n_2+1)$$

$$+34(n_1^2+n_2^2-n_1n_2)+17(n_1+n_2)+18\}+250\{n_1^2+n_2^2+n(m+1)-m\}$$

$$-168m^2+28\Big]$$

After simplification of the expression within the above parenthesis and utilizing $n=n_1+n_2+m+1$, we ultimately derive this third order quota of energy as given below :

$$+\frac{3h^{10}J^{3}(n_1-n_2)n^7}{32(2\pi e)^{10}e\;Z^7\;m_0^5}\Big[23n^2-(n_1-n_2)+11m^2+39\Big] \quad . \quad (26)$$

This tallies with the expression given by Ishida and Hiyama supposed to have been worked out by S Doi, but his calculation is not yet published

We see then the evaluation of only fifth order determinant Δ whose centre is $\theta+a_r$ is quite sufficient to determine the energy-value up to 3rd order of the field-intensity.

7 *Form of the fourth order term*

In this sec we shall lay down the co-efficients of λ'^4 for θ_4 given in sec 4 Remembering the values of the several S's as given in the same sec we write

$$1^\circ\;-\sum_{j,k,l}\frac{S_{rjkl}}{(j-r)(k-r)(l-r)}=-\sum_{s}\frac{(rj)(jr)(kl)(lk)}{(j-r)(k-r)(l-r)}$$

$$+\sum_{\sigma}\frac{(rj)(jk)(kl)(lr)}{(j-r)(k-r)(l-r)}\;,$$

Now

$$\sum\frac{(rj)(jk)(kl)(lr)}{(j-r)(k-r)(l-r)}=\frac{(02)(23)(31)(10)}{2.3\;1.}+\frac{(01)(13)(32)(20)}{1\;3\;2}$$

$$+\frac{(0\bar{2})(\bar{2}\bar{3})(\bar{3}\bar{1})(\bar{1}0)}{-2\ -3\ -1}+\frac{(0\bar{1})(\bar{1}\bar{3})(\bar{3}\bar{2})(\bar{2}0)}{-1\ -3\ -2}+\frac{(01)(1\bar{1})(\bar{1}\bar{2})(\bar{2}0)}{1\ -1\ -2}$$

$$+\frac{(0\bar{1})(\bar{1}1)(12)(20)}{-1\ 1\ 2}+\frac{(02)(21)(1\bar{1})(\bar{1}0)}{2\ 1\ -1}+\frac{(0\bar{2})(\bar{2}\bar{1})(\bar{1}1)(10)}{-2\ -1\ 1}$$

Again

$$-\sum_{j,k,l}\frac{(rj)(jr)(kl)(lk)}{(j-r)(k-r)(l-r)}\equiv\sum_{j,k}\frac{-(rj)(jr)(kj)(jk)}{(j-r)(j-r)(k-r)}$$

$$-\sum_{j\neq k\neq l}\frac{(rj)(jr)(kl)(lk)}{(j-r)(k-r)(l-r)}$$

This latter summation cancels with $\sum_j \frac{S_{rj}}{(j-r)}\ \sum_{\iota,\nu}\frac{S_{\iota\nu}}{(\iota-r)(\nu-r)}$ as this latter is equivalent to $\sum_{j\neq\iota\neq\nu}\frac{(rj)(jr)(\iota\nu)(\nu\iota)}{(j-r)(\iota-r)(\nu-r)}$. The former summation is to be written down next in full Thus

$$-\sum_{j,k}\frac{(rj)(jr)(kj)(jk)}{(j-r)^2(k-r)}=\frac{(01)(10)(13)(31)}{1^2\ 3}+\frac{(0\bar{1})(\bar{1}0)(\bar{1}\bar{3})(\bar{3}\bar{1})}{(-1)^2\ (-3)}$$

$$+\frac{(02)(20)(12)(21)}{2^2\ 1}+\frac{(0\bar{2})(\bar{2}0)(\bar{1}\bar{2})(\bar{2}\bar{1})}{(-2)^2(-1)}+\frac{(01)(10)(12)(21)}{1^2\ 2}$$

$$+\frac{(0\bar{1})(\bar{1}0)(\bar{1}\bar{2})(\bar{2}\bar{1})}{(-1)^2.(-2)}+\frac{(02)(20)(24)(42)}{2^2\ 4}+\frac{(0\bar{2})(\bar{2}0)(\bar{2}\bar{4})(\bar{4}\bar{2})}{(-2)^2\ (-4)}$$

$$+\frac{(01)(10)(1\bar{1})(\bar{1}1)}{1^2.-1}+\frac{(0\bar{1})(\bar{1}0)(\bar{1}1)(1\bar{1})}{(-1)^2\ 1}+\frac{(02)(20)(23)(32)}{2^2.3}$$

$$+\frac{(0\bar{2})(20)(2\bar{3})(32)}{(-2)\ (-3)},$$

$$2^\circ\quad -\sum_{\jmath}\frac{S_{r\jmath}}{(\jmath-r)^2}\;\sum_{\lambda}\frac{S_{r\lambda}}{\lambda(\lambda-r)}=-\sum_{\jmath}\frac{(r\jmath)(\jmath r)}{(\jmath-r)^2}\times\sum_{\lambda}\frac{(r\lambda)(\lambda r)}{(\lambda-r)}=$$

$$-\left[\frac{(01)(10)}{1}+\frac{(02)(20)}{4}+\frac{(0\bar{1})(\bar{1}0)}{1}+\frac{(0\bar{2})(\bar{2}0)}{4}\right]$$

$$\left[\frac{(01)(10)}{1}+\frac{(02)(20)}{2}+\frac{(0\bar{1})(\bar{1}0)}{-1}+\frac{(0\bar{2})(\bar{2}0)}{-2}\right]$$

3°. Values of $S_{r\jmath}(A_\jmath-A_r)^2/(\jmath-r)^3$, $S_{r\jmath k}(A_\jmath-A_r)/(\jmath-r)^2(k-r)$, $S_{r\jmath k}(A_k-A_r)/(\jmath-r)(k-r)^2$ can be written down easily, since the difference of $\jmath$ and r, or $\jmath$ and k cannot exceed $|\,2\,|$, from the form of the determinant

We give below the coefficients of λ'^4 in the proper value θ_4 ;

1st set : calculated from 1° *and* 2°

$$\tfrac{1}{6}[(01)(13)(32)(20)+(02)(23)(31)(10)-(0\bar{1})(\bar{1}\bar{3})(\bar{3}\bar{2})(\bar{2}0)$$

$$-(0\bar{2})(\bar{2}\bar{3})(\bar{3}\bar{1})(\bar{1}0)]+\tfrac{1}{2}[(01)(1\bar{1})(\bar{1}\bar{2})(\bar{2}0)+(0\bar{2})(\bar{2}\bar{1})(\bar{1}1)(10)$$

$$-(\bar{1}0)(02)(21)(1\bar{1})-(\bar{1}1)(12)(20)(0\bar{1})]+\tfrac{1}{16}[(02)(20)(42)(24)$$

$$-(0\bar{2})(\bar{2}0)(\bar{4}\bar{2})(\bar{2}\bar{4})]+\tfrac{1}{12}[(02)(20)(23)(32)-(0\bar{2})(\bar{2}0)(\bar{3}\bar{2})(\bar{2}\bar{3})]$$

$$+\tfrac{1}{4}[(02)(20)(21)(12)-(0\bar{2})(\bar{2}0)(\bar{2}\bar{1})(\bar{1}\bar{2})]+\tfrac{1}{3}[(01)(10)(13)(31)$$

$$-(0\bar{1})(\bar{1}0)(\bar{1}\bar{3})(\bar{3}\bar{1})]+\tfrac{1}{2}[(01)(10)(12)(21)-(0\bar{1})(\bar{1}0)(\bar{1}\bar{2})(\bar{2}\bar{1})]$$

$$+[(0\bar{1})(\bar{1}0)(\bar{1}1)(\bar{1}1)-(01)(10)(1\bar{1})(\bar{1}1)]-\left[\frac{(01)(10)}{1}+\frac{(02)(20)}{4}\right.$$

$$\left.+\frac{(0\bar{1})(\bar{1}0)}{1}+\frac{(0\bar{2})(\bar{2}0)}{4}\right]\left[\frac{(01)(10)}{1}+\frac{(02)(20)}{2}-\frac{(0\bar{1})(\bar{1}0)}{1}\right.$$

$$\left.-\frac{(0\bar{2})(\bar{2}0)}{2}\right]$$

2nd set calculated from 3°.

$$[(01)(10)(A_{r+1}-A_r)^2-(\bar{1}0)(0\bar{1})(A_{r-1}-A_r)^2]$$

$$+\tfrac{1}{8}[(02)(20)(A_{r+2}-A_r)^2-(0\bar{2})(\bar{2}0)(A_{r-2}-A_r)^2]$$

$$+\tfrac{1}{2}[(01)(12)(20)(A_{r+1}-A_r)-(0\bar{1})(\bar{1}\bar{2})(\bar{2}0)(A_{r-1}-A_r)$$

$$+(02)(21)(10)(A_{r+1}-A_r)-(0\bar{2})(\bar{2}\bar{1})(\bar{1}0)(A_{r-1}-A_r)]+$$

$$\tfrac{1}{4}[\{(01)(12)(20)+(02)(21)(10)\}(A_{r+2}-A_r)-\{(0\bar{1})(\bar{1}\bar{2})(\bar{2}0)$$

$$+(0\bar{2})(\bar{2}\bar{1})(\bar{1}0)\}(A_{r-2}-A_r)]+[(01)(1\bar{1})(\bar{1}0)+(0\bar{1})(\bar{1}1)(10)]$$

$$(A_{r-1}-A_{r+1})$$

Translating into the familiar notations β', γ', δ', ϵ', . according to the scheme given in sec 4 we write down the two sets above:—

1st set

$$\tfrac{1}{6}[\beta'_r\delta'_{r+1}\gamma'_{r+2}\epsilon'_{r+3}+\gamma'_r\beta'_{r+1}\epsilon'_{r+2}\delta'_{r+3}-\epsilon'_r\gamma'_{r-1}\delta'_{r-2}\beta'_{r-3}$$

$$-\delta'_r\epsilon'_{r-1}\beta'_{r-2}\gamma'_{r-3}]$$

$$+\tfrac{1}{2}[\epsilon'_r\delta'_{r+1}\beta'_{r-1}\gamma'_{r-2}+\gamma'_r\epsilon'_{r+1}\delta'_{r-1}\beta'_{r-2}-\delta'_r\epsilon'_{r+2}\gamma'_{r+1}\beta'_{r-1}$$

$$-\epsilon'_{r+1}\delta'_{r+2}\beta'_r\gamma'_{r-1}]$$

$$+\tfrac{1}{16}[\epsilon'_{r+2}\beta'_r\beta'_{r+2}\epsilon'_{r+4}-\beta'_{r-2}\epsilon'_r\epsilon'_{r-2}\beta'_{r-4}]+\tfrac{1}{12}[\epsilon'_{r+2}\beta'_r\delta'_{r+3}\gamma'_{r+2}$$

$$-\beta'_{r-2}\epsilon'_r\delta'_{r-2}\gamma'_{r-3}]$$

$$+\tfrac{1}{4}[\epsilon'_{r+2}\beta'_r\gamma'_{r+1}\delta'_{r+2}-\beta'_{r-2}\epsilon'_r\delta'_{r-1}\gamma'_{r-2}]+\tfrac{1}{3}[\delta'_{r+1}\gamma'_r\epsilon'_{r+3}\beta'_{r+1}$$

$$-\gamma'_{r-1}\delta'_r\beta'_{r-3}\epsilon'_{r-1}]$$

$$+[\gamma'_{r-1}\delta'_r\beta'_{r-1}\epsilon'_{r+1}-\delta'_{r+1}\gamma'_r\beta'_{r-1}\epsilon'_{r+1}]-\Bigg[\frac{\gamma'_r\delta'_{r+1}}{1}+\frac{\epsilon'_{r+2}\beta'_r}{4}$$

$$+\frac{\gamma'_{r-1}\delta'_r}{1}+\frac{\beta'_{r-2}\epsilon'_r}{4}\Bigg]\times\Bigg[\frac{\gamma'_r\delta'_{r+1}}{1}+\frac{\epsilon'_{r+2}\beta'_r}{2}-\frac{\gamma'_{r-1}\delta'_r}{1}$$

$$-\frac{\beta'_{r-2}\epsilon'_r}{2}\Bigg].$$

2nd set.

$$[\delta'_{r+1}\gamma'_r(A_{r+1}-A_r)^2-\gamma'_{r-1}\delta'_r(A_{r-1}-A_r)^2]+\tfrac{1}{5}[\epsilon'_{r+2}\beta'_r(A_{r+2}-A_r)^2$$
$$-\beta'_{r-2}\epsilon'_r(A_{r-2}-A_r)^2]$$
$$+\tfrac{1}{2}[(\delta'_{r+1}\delta'_{r+2}\beta'_r+\epsilon'_{r+2}\gamma'_{r+1}\gamma'_r)(A_{r+1}-A_r)-(\gamma'_{r-1}\gamma'_{r-2}\epsilon'_r$$
$$+\beta'_{r-2}\delta'_{r-1}\delta'_r)(A_{r-1}-A_r)]$$
$$+\tfrac{1}{4}[(\delta'_{r+1}\delta'_{r+2}\beta'_r+\epsilon'_{r+2}\gamma'_{r+1}\gamma'_r)(A_{r+2}-A_r)-(\gamma'_{r-1}\gamma'_{r-2}\epsilon'_r$$
$$+\beta'_{r-2}\delta'_r\delta'_{r-1}(A_{r-2}-A_r)]$$
$$+[\delta'_{r+1}\beta'_{r-1}\delta'_r+\gamma'_{r-1}\epsilon'_{r+1}\gamma'_r](A_{r-1}-A_{r+1})$$

Another method of arriving at the result is given in Appendix I

8 *Calculation of 1st set*

Coefficients of the 1st set can be calculated with the least chance of committing any mistake if we stick to the following scheme:—

$$\Delta(r)\equiv r(r+m)(2r+m),\quad F(r)\equiv r(r+m)\Delta(r-1)\Delta(r+1),$$

$$\psi(r)\equiv 4(2r+m+2)\Delta(r+1),\quad \theta(r)\equiv(r+1)(r+2)(r+m+1)(r+m+2)$$

$$\text{Set I}=\tfrac{5}{8}[F(r+2)-F(r-1)]+4[F(r)-F(r+1)]+$$
$$\tfrac{1}{8}[\psi(r)\psi(r+1)-\psi(r-1)\psi(r-2)]-$$
$$[\psi(r)+\psi(r-1)][\psi(r)-\psi(r-1)]+$$
$$\tfrac{1}{16}[\theta(r)\theta(r+2)-\theta(r-2)\theta(r-4)]-$$
$$\tfrac{1}{8}[\theta^2(r)-\theta^2(r-2)]+\tfrac{1}{12}[\theta(r)\psi(r+2)-\theta(r-2)\psi(r-3)]$$
$$+\tfrac{1}{4}[\theta(r)\psi(r+1)-\theta(r-2)\psi(r-2)]+\tfrac{1}{8}[\psi(r)\theta(r+1)-\psi(r-1)\theta(r-3)]$$
$$+[\psi(r-1)-\psi(r)][\theta(r-1)]-\tfrac{1}{4}[\theta(r)][\psi(r)-\psi(r-1)]$$
$$-\tfrac{1}{2}[\theta(r)][\psi(r)+\psi(r-1)]-\tfrac{1}{4}[\theta(r-2)][\psi(r)+\psi(r-1)]$$
$$+\tfrac{1}{2}[\theta(r-2)][\psi(r)+\psi(r-1)]$$

The above can be rearranged thus :

$$\text{Set I}=\{I\}+\{II\}+\{III\}+\{IV\},$$

where $\{I\}\equiv\tfrac{5}{8}[F(r+2)-F(r-1)]+4[F(r)-F(r+1)]$,

$$\{II\} \equiv \tfrac{1}{2}[\psi(r)\psi(r+1)-\psi(r-1)\psi(r-2)]$$

$$-[\psi(r)+\psi(r+1)][\psi(r)-\psi(r-1)]\,,$$

$$\{III\} \equiv \tfrac{1}{16}[\theta(r)\theta(r+2)-\theta(r-2)\theta(r-4)]-\tfrac{1}{8}[\theta^2(r)-\theta^2(r-2)]\,,$$

$$\{IV\} \equiv \tfrac{1}{12}[\theta(r)\psi(r+2)-\theta(r-2)\psi(r-3)]$$

$$+\tfrac{1}{4}[\theta(r)\psi(r+1)-\theta(r-2)\psi(r-2)]$$

$$+\tfrac{1}{3}[\psi(r)\theta(r+1)-\psi(r-1)\theta(r-3)]$$

$$-[\psi(r)-\psi(r-1)][\theta(r-1)]$$

$$+\tfrac{1}{2}[\theta(r)-\theta(r-2)][\psi(r)+\psi(r-1)]$$

$$-\tfrac{1}{4}[\theta(r)+\theta(r-2)][\psi(r)-\psi(r-1)]$$

We derive the following results after some laborious calculations and reductions :—

$$\{I\}=4[448r^5+1120(m+1)r^4+(1000m^2+2240m+1760)r^3+(380m^3$$
$$+1500m^2+2640m+1520)r^2+(56m^4+380m^3+1164m^2+1520m$$
$$+720)r+(2m^5+28m^4+142m^3+332m^2+360m+144)]\,,$$

$$\{II\}=8[1152r^5+2880(m+1)r^4+(2656m^2+5760m+3968)r^3$$
$$+(1104m^3+3984m^2+5952m+3072)r^2+(196m^4+1104m^3$$
$$+2736m^2+3072m+1280)r+(10m^5+98m^4+376m^3+704m^2$$
$$+640m+224)]\,,$$

$$\{III\}=36r^5+90(m+1)r^4+(76m^2+180m+212)r^3+(24m^3+114m^2$$
$$+318m+228)r^2+(2m^4+24m^3+130m^2+228m+138)r+(m^4$$
$$+12m^3+46m^2+69m+34)\,,$$

$$\{IV\}=2048r^5+5120(m+1)r^4+(4544m^2+10240m+8576)r^3$$
$$+(1696m^3+6816m^2+12864m+7744)r^2+(240m^4+1696m^3$$
$$+5600m^2+7744m+3904)r+(8m^5+120m^4+656m^3+1664m^2$$
$$+1952m+832)$$

The method of arriving at the above results is given in APPENDIX II.

9 *Calculation of the 2nd set*

The 2nd set can be arranged thus

Set II $\equiv \{V\}+\{VI\}+\{VII\}$,

where

$$\{V\}=\delta'_{r+1}\gamma'_r(A_{r+1}-A_r)^2-\delta'_r\gamma'_{r-1}(A_{r-1}-A_r)^2 ,$$

$$\{VI\}=\tfrac{1}{8}[\epsilon'_{r+2}\beta'_r(A_{r+2}-A_r)^2-\epsilon'_r\beta'_{r-2}(A_{r-2}-A_r)^2],$$

$\{VII\}$ is of the form $u_2A_{r+2}+u_1A_{r+1}+u_0A_r+u_{-1}A_{r-1}+u_{-2}A_{r-2}$

We derive the following results after some calculations, the details of which are given in APPENDIX III

$$\{V\}=144[\psi(r)\ (2r+m+2)^2-\psi(r-1)\ (2r+m)^2]$$

$$=144[96r^5+240(m+1)r^4+(224m^2+480m+320)r^3+$$
$$(96m^3+336m^2+480m+240)r^2+(18m^4+96m^3+224m^2$$
$$+240m+96)r+(m^5+9m^4+32m^3+56m^2+48m+16)],$$

$$\{VI\}=18[\theta(r)(2r+m+3)^2-\theta(r-2)(2r+m-1)^2]$$

$$=18[48r^5+120(m+1)r^4+(104m^2+240m+264)r^3$$
$$+(36m^3+156m^2+396m+276)r^2+(4m^4+36m^3+168m^2$$
$$+276m+156)r+(2m^4+18m^3+58m^2+78m+36)],$$

and $\{VII\}$ is the sum of five expressions given below —

1°.. $(A_{r-1}-A_{r+1})(\delta'_{r+1}\beta'_{r-1}\delta'_r+\gamma'_{r-1}\epsilon'_{r+1}\gamma'_r\quad)=$

$$-96r(r+1)(r+m)(r+m+1)(2r+m)(2r+m+1)(2r+m+2)=$$
$$-96\theta(r-1)\ (2r+m)(2r+m+1)(2r+m+2),$$

2° $\tfrac{1}{4}(A_{r+2}-A_r)(\delta'_{r+1}\delta'_{r+2}\beta'_r+\epsilon'_{r+2}\gamma'_{r+1}\gamma'_r)=$

$$24(r+1)(r+2)(r+m+1)(r+m+2)(2r+m+2)(2r+m+3)(2r+m+4)$$
$$=24\theta(r)\ (2r+m+2)(2r+m+3)(2r+m+4);$$

$3°\quad \frac{1}{4}(A_r - A_{r-2})(\gamma'_{r-1}\gamma'_{r-2}\epsilon'_r + \beta'_{r-2}\delta'_r\delta'_{r-1}) =$

$24r(r-1)(r+m)(r+m-1\ (2r+m)(2r+m-1)(2r+m-2)$

$= 24\theta(r-2)\ (2r+m)(2r+m-1)(2r+m-2)\ ,$

$4°\ .\ \frac{1}{2}(A_{r+1} - A_r)(\delta'_{r+1}\delta'_{r+2}\beta'_r + \epsilon'_{r+2}\gamma'_{r+1}\gamma'_r) =$

$24(r+1)(r+2)(r+m+1)(r+m+2)(2r+m+2)^2(2r+m+4)$

$= 24\theta(r).(2r+m+2)^2(2r+m+4\ \ ;$

$5°\ .\ \frac{1}{2}(A_r - A_{r-1})(\gamma'_{r-1}\gamma'_{r-2}\epsilon'_r + \beta'_{r-2}\delta'_r\delta'_{r-1}) =$

$24r(r-1)(r+m)(r+m-1)(2r+m-2)(2r+m)^2$

$= 24\theta(r-2)\ (2r+m-2)(2r+m)^2\ ,$

Adding up 1°, 2°, , 5° we get

$\{VII\} = 24[624r^5 + 1560(m+1)r^4 + (1416m^2 + 3120m + $
$+ 2304)r^3 + (564m^3 + 2124m^2 + 3456m + 1896)r^2 + (92m^4$
$+ 564m^3 + 1556m^2 + 1896m + 848)r + (4m^5 + 46m^4 + 202m^3$
$+ 424m^2 + 424m + 160)]$

10 *Calculation of the 4th order effect*

Coefficient of λ'^4 is the total sum of the sets I and II, which is the grand total of $\{I\}$, $\{II\}$, . .$\{VII\}$ Let this be denoted by $H(r)$

Thus

$$H(r) = 42756r^5 + 106890(m+1)r^4 + (97980m^2 + 213780m + 153700)r^3 + (40080m^3 + 146970m^2 + 230550m + 123660)r^2 + (6906m^4 + 40080m^3 + 104898m^2 + 123660m + 54146)r + (336m^5 + 3453m^4 + 14024m^3 + 27954m^2 + 27073m + 10026) \quad .. \quad (27)$$

As we have done in sec 6 in the case of 3rd order effect we write down

$$\frac{\pi^2 m_0}{h^2\sqrt{-A}}(Ze^2 - \beta) - \frac{1}{2}(m+1) = n_1 + \lambda'(\quad) - \lambda'^2(\quad) + \lambda'^3(\quad) - \lambda'^4(H(n_1)),$$

$$\frac{\pi^2 m_0}{h^2\sqrt{-A}}(Ze^2 + \beta) - \frac{1}{2}(m+1) = n_2 - \lambda'(\quad) - \lambda'^2(\quad) - \lambda'^3(\quad) - \lambda'^4(H(n_2)).$$

Adding up both sides of the above and transposing we get,

$$\frac{2\pi^2 m_0 Ze^2}{h^2\sqrt{-A}} = n\left\{1+\lambda\frac{g_1}{(2\sqrt{-A})^3}-\lambda^2\frac{g_2}{(2\sqrt{-A})^6}\right.$$

$$\left.+\lambda^3\frac{g_3}{(2\sqrt{-A})^9}-\lambda^4\frac{g_4}{(2\sqrt{-A})^{12}}\right\} \tag{28}$$

where the values of g_1, g_2, g_3 are the same as given in sec 6, and g_4 will be the coefficient of $n(=n_1+n_2+m+1)$ in the expression for $H(n_1)+H(n_2)$ We give below the values of these g's

$$g_1 \equiv 6(n_1-n_2),$$

$$g_2 \equiv 2[4m^2+17(n_1+n_2+1)m+34(n_1^2+n_2^2-n_1n_2)+17(n_1+n_2)+18]$$

$$\equiv 2[34n^2-51(m+1)n-102n_1n_2+21m^2+51m+35],$$

$$g_3 \equiv 6(n_1-n_2)[250(n_1^2+n_2^2)+250(m+1)n-168m^2-250m+28]$$

$$\equiv 6(n_1-n_2)[250n^2-250(m+1)n-500n_1n_2+82m^2+250m+278],$$

$$ng_4 \equiv H(n_1)+H(n_2)$$

$$\equiv 2n[21378n^4-53445(m+1)n^3+(48990m^2+106890m+76850)n^2$$

$$-(20040m^3+73485m^2+115275m+61830)n+(3453m^4+20040m^3$$

$$+52449m^2+61830m+27073)-106890n^2n_1n_2+106890(m+1)nn_1n_2$$

$$+106890n_1^2n_2^2-(40080m^2+106890m+123660)n_1n_2],$$

obtained after factorization

We have given in the above, two equivalents for g_2 and g_3 instead of one (as given in sec 6) as the new values will be useful for our future calculation

We shall follow the usual course of successive approximations for calculation of energy We shall denote by K_4 the value of K (given in sec 4, equa (24)) for the 4th order approximation that is to say, if

$$K=c/n\left[1+\lambda\frac{g_1}{(2\sqrt{-A})^3}-\lambda^2\frac{g_2}{(2\sqrt{-A})^6}\right.$$

$$\left.+\lambda^3\frac{g_3}{(2\sqrt{-A})^9}-\lambda^4\frac{g_4}{(2\sqrt{-A})^{12}}\right] \tag{29}$$

then

$$K_4 = c\,\blacksquare\, n \left[1 + \lambda \frac{g_1}{K_3^3} - \lambda^2 \frac{g_2}{K_2^6} + \lambda^3 \frac{g_3}{K_1^9} - \lambda^4 \frac{g_4}{K_0^{12}} \right]$$

$$= c\,\blacksquare\, n \left[1 + \lambda \frac{n^3}{c^3} g_1 + \lambda^2 \frac{n^6}{c^6} (3g_1^2 - g_2) + \lambda^3 \frac{n^9}{c^9} (12g_1^3 - 9g_1 g_2 + g_3) \right.$$

$$\left. + \lambda^4 \frac{n^{12}}{c^{12}} (55g_1^4 - 66g_1^2 g_2 + 12g_1 g_3 + 6g_2^2 - g_4) \right], \quad \ldots \quad (30)$$

where the coefficient of λ^4 has been found out after an easy algebraical calculation

Next put $2\sqrt{-A} = K_4$, and square both sides. We get

$$-4A = c^2 \blacksquare\, n^2 \left[1 - 2\lambda \frac{n^3}{c^3} g_1 + \lambda^2 \frac{n^6}{c^6} (2g_2 - 3g_1^2) - \lambda^3 \frac{n^9}{c^9} (10g_1^3 \right.$$

$$\left. - 12g_1 g_2 + 2g_3) + \lambda^4 \frac{n^{12}}{c^{12}} (2g_4 + 72g_1^2 g_2 - 42g_1^4 - 18g_1 g_3 - 9g_2^2) \right] \quad (31)$$

obtained by an easy algebraical expansion in Binomial Theorem

Remembering $c = 4\pi^2 m_0 Ze^2/h^2$, $A = 2\pi^2 m_0 E/h^2$, $\lambda = \pi^2 m_0 eJ/h^2$ it is now easy to set down the 4th order quota of energy. Supposing $E = E_0 + E_1 + E_2 + E_3 + E_4$ where E_1, E_2 involve respectively 1st, 2nd, .. powers of λ or J, as multiplier, we obtain :

$$E_4 = -\frac{h^{14} J^4 n^{10}}{2^9 (2\pi e)^{14} m_0^7 Z^{10} e^2} \left[2g_4 + 72g_1^2 g_2 - 18g_1 g_3 - 9g_2^2 - 42g_1^4 \right] \cdot \quad (32)$$

The expression within parenthesis of the above $= 4\,T(n, m)$ say, where

$$T(n, m) \equiv 930n^4 - 525(m+1)n^3 + (-231m^2 + 672m + 5453)n^2$$

$$-(-210m^3 - 63m^2 + 1659m + 1512)n + (-192m^4 - 210m^3$$

$$+ 12840m^2 + 1512m + 3664) + 294n_1^2 n_2^2 - 1050n^2 n_1 n_2$$

$$+ 294(m+1)nn_1 n_2 - (-420m^2 + 294m + 3024)n_1 n_2 \quad \ldots \quad (33)$$

Calculations leading to the above result from the values of g_1, g_2, g_3, g_4 are given in APPENDIX IV

Finally, the term-value for hydrogen ($Z=1$) is given below:

$$h\nu/c = -\frac{h^{13} J^4 n^{10}}{2^{21}\, \pi^{14}\, m_0^7 e^{16}}\, T(n,m). \quad \ldots \quad (34)$$

11. *Stark-effect shift of the Balmer lines*

The shift in question can be found out from the frequency-change $\Delta\nu$ If ν is expressed in wave numbers and the field-strength J in Volt/cm, we write

$$\Delta\tilde{\nu}=A(\ )_i^f J-B(\ )_i^f J^2+C(\ )_i^f J^3-D(\ )_i^f J^4 \quad \dots \quad (35)$$

where the expressions within parentheses above involve g_1, g_2, g_3, g_4 as linear, quadratic, cubic and bi-quadratic in them respectively, and as difference for the two quantum states—initial and final The values of the coefficients A, B, C, D are given below —

$$A=\frac{3h}{8\pi^2 m_0 ec(300)}=6,439\ 10^{-5},$$

$$B=\frac{h^5}{2^{10}.\ \pi^6\ m_0^3\ e^6\ c(300)^2}=5,2995\ 10^{-16},$$

$$C=\frac{3h^9}{2^{15}\ \pi^{10} m_0^5\ e^{11}\ c(300)^3}=1,568\ 10^{-25},$$

$$D^*=\frac{h^{13}}{2^{21}\ \pi^{14}\ m_0^7\ e^{16}\ c(300)^4}=2,5746\ 10^{-36},$$

calculated from the following values of physical constants —

$h=6,55\times10^{-27}$(erg /sec), $e=4,77\times10^{-10}$ (e s u),

$m_0=8,99\times10^{-28}$(gm), $c=3\times10^{10}$(cm /sec)

Briefly equa (35) can be written in the form

$$\Delta\tilde{\nu}=aJ-bJ^2+cJ^3-dJ^4 \qquad . \quad (36)$$

* Values of A, B, C have been taken from Ishida and Hiyama's paper (l c). Calculation of D is given here —

$$D=\frac{(6,55)^{13}\ (10^{-27})^{13}}{2^{21}\ (3.1416)^{14}\ (8,99)^7\ (10^{-28})^7\ (4,77)^{16}\ (10^{-10})^{16}\ (3\ 10^{10})\ 3^4\ 10^8}$$

$$=\frac{(6,55)^{13}\ 10^{-13}}{2^{21}\ (3,1416)^{14}.\ (8,99)^7\ (4,77)^{16}\ 3^5},$$

$\log D=-13+13\log 6,55-21\log 2-14\log 3,1416-7\log 8,99-16\log 4,77-5\log 3$

$=-13+13\ (0,8162)-21\ (0,30103)-14\ (0,4972)-7\ (0,9538)$

$-16\ (0,6785)-5\ (0,4771)=-13+10,6106-6,3210-6,9608$

$-6,6766-10,8560-2,3855=-13+10,6106-33,1999=\overline{36},4107$

Hence $D=2,5746\times10^{-36}$.

The values of the constants a, b, c have been given by Ishida and Hiyama for J=1 Million Volt/cm for the several π—and σ—components of H_α, H_β and H_γ, and I have calculated the values of d for the same components and field strength These values are given in Table I Details of calculation are given in APPENDIX V

TABLE I

Balmer Lines	Pol	Combination	a	b	c	d
H_α	π	(111)(011) $\Delta=2$	128, 78	6, 715	0, 003	1, 485 10^{-2}
H_α	π	(102)(002) $\Delta=3$	193, 17	6, 207	0, 088	1, 642 10^{-2}
H_α	π	(201)(101) $\Delta=4$	257, 56	6, 309	0, 164	1, 649 10^{-2}
H_α	σ	(003)(002) $\Delta=0$	0	5, 177	0	2, 032 10^{-2}
H_α	σ	(111)(002) $\Delta=0$	0	6, 705	0	1, 449 10^{-2}
H_α	σ	(102)(101) $\Delta=1$	64, 39	6, 156	0, 085	1, 647 10^{-2}
H_β	π	(211)(011) $\Delta=6$	386, 34	38, 36	1, 045	0, 683
H_β	π	(202)(002) $\Delta=8$	515, 12	35, 97	2, 125	0, 742
H_β	π	(301)(101) $\Delta=10$	643, 90	35, 10	3, 065	0, 686
H_β	σ	(112)(011) $\Delta=2$	128, 78	37, 54	0, 003	0, 729
H_β	σ	(103)(002) $\Delta=4$	257, 56	33, 53	1, 155	0, 735
H_β	σ	(211)(002) $\Delta=4$	257, 56	38, 41	1, 042	0, 772
H_β	σ	(202)(011) $\Delta=6$	386, 34	35, 92	2, 125	0, 942
H_γ	π	(221)(011) $\Delta=2$	128, 78	146, 3	0, 003	13, 59
H_γ	π	(212)(002) $\Delta=5$	321, 95	142, 2	7, 65	13, 72
H_γ	π	(311)(011) $\Delta=12$	772, 68	142, 5	14, 92	14, 71
H_γ	π	(302)(002) $\Delta=15$	965, 85	134, 2	22, 64	15, 293
H_γ	π	(401)(101) $\Delta=18$	1159, 02	130, 5	29, 30	16, 26
H_γ	σ	(113)(002) $\Delta=0$	0	134, 1	0	13, 73
H_γ	σ	(221)(002) $\Delta=0$	0	146, 3	0	13, 61
H_γ	σ	(212)(101) $\Delta=3$	193, 17	142, 2	7, 65	13, 77
H_γ	σ	(203)(002) $\Delta=10$	643, 90	130, 5	16, 00	14, 58
H_γ	σ	(311)(002) $\Delta=10$	643, 90	142, 5	14, 93	14, 26
H_γ	σ	(302)(101) $\Delta=13$	837, 07	134, 3	22, 64	15, 06

It may be noted that the values of a will be the same in both old and new quantum mechanics, but values of b and c will be different

The above Table enables us not only to calculate the theoretical Stark-effect shift for the Balmer lines H_α, H_β, H_γ for field-strengths in Million-Volts/cm. but also to draw a graph showing the deviation of the theoretical Stark-effect shift from the observed Stark-effect shift The following are the wave-lengths of the Balmer lines H_α, H_β, H_γ, H_δ, as accepted by the International Board of spectroscopists after taking means of such doublet lines —

$H_\alpha = 6562,793$ Å, $H_\beta = 4861,327$ Å, $H_\gamma = 4340,466$ Å, $H_\delta = 4101,738$ Å, where

$$\tilde{\nu} = N \left[\frac{1}{2^2} - \frac{1}{(n+\mu)^2} \right],$$

$n=3, 4, 5$ for H_α, H_β, H_γ respectively, and $N=109679{,}22$, $\mu \equiv 6{,}9\ 10^{-6}$,

The Table II annexed here gives $\Delta\tilde{\nu}$ as a function of J, as the field increases from 0, 6 Million Volt/cm to 1 Million Volt/cm, for the case of H_γ—line (π 18), which has been discussed by Gebauer & Rausch (l c), when $\Delta\lambda$ is expressed in Angstrom unit The last column has been added by me after calculation for the 4th order effect, the details of which are given in APPENDIX VI

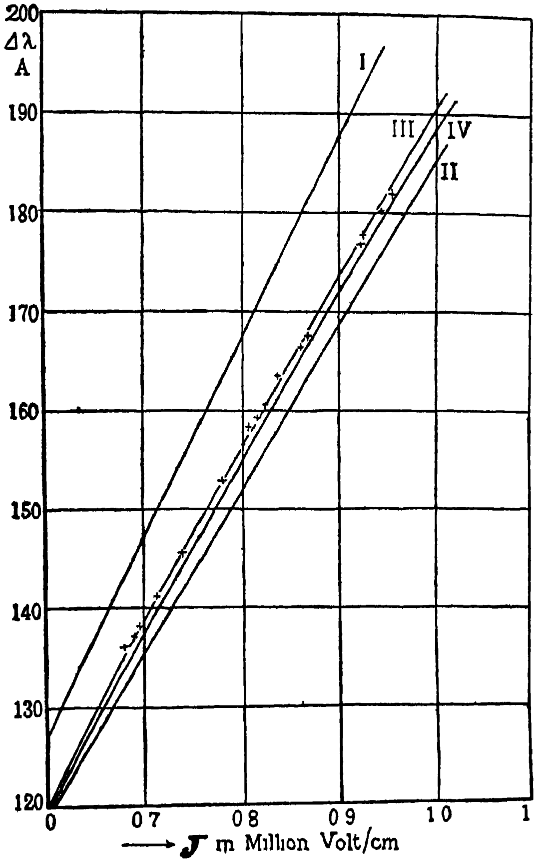
TABLE II

J	$\Delta\tilde{\nu}=$		(in $\Delta\lambda$ calculated)	
Volt/cm.	aJ	$aJ-bJ^2$	$aJ-bJ^2+cJ^3$	$aJ-bJ^2+cJ^3-dJ^4$
	Å	Å	Å	Å
600000	127,17	118,82	119,95	119,57
700000	147,65	136,88	138,16	134,44
800000	167,98	153,33	155,97	154,80
900000	188,01	169,70	173,42	171,56
1000000	207,90	185,49*	190,55	188,47

From the data given in the above Table II a $\Delta\lambda$—J—graph, may be entitled the "deviation graph," has been drawn, showing the departure of calculated $\Delta\lambda$ from the actual $\Delta\lambda$, based on experiment

* The figure is 184,57 according to Gebauer & Rausch, ZS f Phys., 62, 1930,

The experimental curve is not drawn in the graph but an alignment of its course can be noticed from points indicated by (+) crossed signs These experimental values are taken from the work of Gebauer & Rausch (l c) Altogether four theoretical curves are drawn, of which curves for the first three Stark-effect orders were drawn by them and the curve IV has been drawn by me from my data for the 4th order effect Curve IV drawn by me deviates a little more from the experimental than curve III drawn by them This points to indicate that the perturbation calculation up to the 3rd order is just sufficient for this line H_γ π 18



12. *Conclusion and Discussion*

The determinant $\Delta[\theta]$ given in sec 3 for calculation of eigen-values is not a convergent determinant of the normal type as discussed by Koch (l c) because when the leading diagonal terms have been made unity by dividing each row by the corresponding element of the leading diagonal the Kochian condition of convergency $\sum_{i=0}^{\infty} \sum_{k=0}^{8} | A_{i,k} |$ as given in sec 2 is not satisfied when r increases indefinitely, and this is true however small the value of the perturbation parameter λ, that is to say, however small the field strength J For very weak field

the perturbation calculation may be carried on to a higher order than for strong field And as we know that for weak field 2nd order effect gives good results in general probably, in this case, 3rd and 4th order terms give better results more approaching actual—if the quantum number is not large, (i e, in the case of hydrogen or hydrogenic atoms)—for the H_α and H_β lines than for the H_γ and H_δ lines For strong field 1st order effect gives good results in general and the result would be worse if one proceeds from lower to higher quantum numbers, i e, from H_α and H_β to H_γ and H_δ These are our conclusions from theory which were not clear from either the old or new perturbation theory as presented by previous workers headed by Schrodinger

Table I shows that except for certain components of H_β H_γ the values of d are greater than the corresponding values of c as given by Ishida & Hiyama This means that $\Delta\tilde{\nu}$ cannot be expressed as an ascending series of the field strength for 1 Million-Volt to a great extent, and we have seen that in the case of H_γ π 18 the series is limited up to 3rd order, for the 4th order calculation leads us a bit far from the actual

Further, if we divide each row by the corresponding ϵ-term which involves the highest (i e, 4th degree) term in quantum number r, we get, of course, an *einheit*-determinant, but the leading diagonal terms are not unity in this new form of the determinantal equation, and the method of expanding such determinants is not known to me to have been discussed by any pure mathematician up till now At any rate, the expansion cannot lead us to a convergent series in powers of J, and the perturbation theory has, therefore, some limitations, in quantum or Schrodinger mechanics

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APPENDIX I

If we put $\rho=2\sqrt{-A}\,r$ in equation (7) we get

$$f''+\frac{1}{\rho}f'+\left(-\frac{1}{4}+\frac{B}{\sqrt{-A}}\frac{1}{\rho}-\frac{m^2}{4\rho^2}\right)f=\lambda'\rho f \qquad (7)$$

Make this self-adjoint by multiplying it by ρ Thus

$$\frac{d}{d\rho}(\rho f')+\left[\eta-\frac{\rho}{4}-\frac{m^2}{4\rho}\right]f=\lambda'\rho^2 f \quad \ldots \quad (7')$$

where $\eta=\frac{B}{\sqrt{-A}}$ $\left[\eta_0=n, \text{ and } f_0=e^{-\frac{\rho}{2}}\rho^{\frac{m}{2}}L^m_{n+m}(\rho)\right]$,

Put

$$f=f_0+\lambda' f_1+\lambda'^2 f_2+\lambda'^3 f_3+\ldots,$$

$$\eta=\eta_0+\lambda'\eta_1+\lambda'^2\eta_2+\lambda'^3\eta_3+\ldots,$$

and equate the various powers of λ' to zero, so that we derive a system of equations:—

$$\frac{d}{d\rho}(\rho f'_0)+\left(\eta_0-\frac{\rho}{4}-\frac{m^2}{4\rho}\right)f_0=0, \quad \ldots \quad (7.1)$$

$$\frac{d}{d\rho}\rho f'_1)+\left(\eta_0-\frac{\rho}{4}-\frac{m^2}{4\rho}\right)f_1=(\rho^2-\eta_1)f_0, \quad \ldots \quad (7.2)$$

$$\frac{d}{d\rho}(\rho f'_2)+\left(\eta_0-\frac{\rho}{4}-\frac{m^2}{4\rho}\right)f_2=(\rho^2-\eta_1)f_1-\eta_2 f_0, \quad \ldots \quad (7.3)$$

$$\frac{d}{d\rho}(\rho f'_3)+\left(\eta_0-\frac{\rho}{4}-\frac{m^2}{4\rho}\right)f_3=(\rho^2-\eta_1)f_2-\eta_2 f_1-\eta_3 f_0, \qquad (7.4)$$

$$\frac{d}{d\rho}(\rho f'_4)+\left(\eta_0-\frac{\rho}{4}-\frac{m^2}{4\rho}\right)f_4=(\rho^2-\eta_1)f_3-\eta_2 f_2-\eta_3 f_1-\eta_4 f_0 \qquad (7.5)$$

... ...

$$\frac{d}{d\rho}(\rho f'_k)+\left(\eta_0-\frac{\rho}{4}-\frac{m^2}{4\rho}\right)f_k=(\rho^2-\eta_1)f_{k-1}$$

$$-\eta_2 f_{k-2}-\eta_3 f_{k-3}- \quad -\eta_k f_0 \qquad \ldots \quad (7\ k+1)$$

Now $\rho^2 f_0=\rho^2 e^{-\frac{\rho}{2}}\rho^{\frac{m}{2}}\,L^m_{m+n}(\rho)\equiv\rho^2 g(\rho)L^m_\nu(\rho)$ say, where

$$g(\rho)\equiv e^{-\frac{\rho}{2}}\ \rho^{\frac{m}{2}},\ \nu=m+n$$

$$\therefore\ \rho^2 f_0=g(\rho)[\beta'_n L^m_{\nu+2}+\gamma'_n L^m_{\nu+1}+A_n L^m_\nu+\delta'_n L^m_{\nu-1}+\epsilon'_n L^m_{\nu=2}],$$

where

$$\beta'_n\equiv\frac{(n+1)(n+2)}{(\nu+1)(\nu+2)}\ ,\ \gamma'_n\equiv-\frac{2(n+1)(2n+m+2)}{(\nu+1)},$$

$$A_n\equiv 6n^2+6nm+6n+m^2+3m+2\,,$$

$$\delta'_n\equiv-2\nu^2(2n+m),\ \epsilon'_n\equiv\nu^2(\nu-1)^2 \qquad cf \text{ equa } (11)$$

For calculation of perturbations for different orders I have followed Schrödinger's method as delineated in Sommerfeld's "Wave mechanics,"[2] Chap II, sec 2, and dealt with *eigenfunctions* fully

1*st order perturbation*

$$\eta_1=\frac{\int_0^\infty \rho^2 f_0^2 d\rho}{\int_0^\infty f_0^2 d\rho}=\frac{\int_0^\infty (g(\rho))^2 L^m_\nu[\beta'_n L^m_{\nu+2}+\ldots+\epsilon'_n L^m_{\nu-2}]d\rho}{\int_0^\infty (g(\rho))^2\ (L^m_\nu(\rho))^2 d\rho}$$

$$=A_n\ \frac{\int_0^\infty (g(\rho))^2 (L^m_\nu(\rho))^2 d\rho}{\int_0^\infty (g(\rho))^2 (L^m_\nu(\rho))^2 d\rho}=A_n \qquad .. \quad (8)$$

from condition of orthogonality of associated Laguerre polynoms

Suppose $f_1=g(\rho)[BL^m_{\nu+1}+CL^m_{\nu+1}+AL^m_\nu+DL^m_{\nu-1}+EL^m_{\nu-2}],$

where the coefficients B, C,...can be calculated from (7 2), by substitution Thus

$$B\frac{d}{d\rho}\left[\rho\frac{d}{d\rho}\{g(\rho)L^m_{\nu+2}\}\right]+\left[\left(\eta_0-\frac{\rho}{4}-\frac{m^2}{4\rho}+2\right)-2\right][g(\rho)L^m_{\nu+2}]B+$$

$$C\frac{d}{d\rho}\left[\rho\frac{d}{d\rho}\{g(\rho)L^m_{\nu+1}\}\right]+\left[\left(\eta_0-\frac{\rho}{4}-\frac{m^2}{4\rho}+1\right)-1\right][g(\rho)L^m_{\nu+1}]C+$$

$$A\frac{d}{d\rho}\left[\rho\frac{d}{d\rho}\{g(\rho)L^m_{\nu}\}\right]+\left[\left(\eta_0-\frac{\rho}{4}-\frac{m^2}{4\rho}\right)\right][g(\rho)L^m_{\nu}]A+$$

$$D\frac{d}{d\rho}\left[\rho\frac{d}{d\rho}\{g(\rho)L^m_{\nu-1}\}\right]+\left[\left(\eta_0-\frac{\rho}{4}-\frac{m^2}{4\rho}-1\right)+1\right][g(\rho)L^m_{\nu-1}]D+$$

$$E\frac{d}{d\rho}\left[\rho\frac{d}{d\rho}\{g(\rho)L^m_{\nu-2}\}\right]+\left[\left(\eta_0-\frac{\rho}{4}-\frac{m^2}{4\rho}-2\right)+2\right][g(\rho)L^m_{\nu-2}]E$$

$$=(\rho^2-\eta_1)g(\rho)L^m_{\nu}(\rho)$$

$$=g(\rho)[\beta'_n L^m_{\nu+2}+\gamma'_n L^m_{\nu+1}+(A_n-\eta_1)L^m_{\nu}+\delta'_n L^m_{\nu-1}+\epsilon'_n L^m_{\nu-2}]$$

Equating coefficients of $L^m_{\nu+2}$,...from both sides we get

$$-2B=\beta'_n,\ -B=\gamma'_n,\ A=A_n-\eta_1=0,\ D=\delta'_n,\ 2E=\epsilon'_n$$

$$f_1=g(\rho)[-\tfrac{1}{2}\beta'_m L^m_{\nu+2}-\gamma'_n L^m_{\nu+1}+\delta'_n L^m_{\nu-1}+\tfrac{1}{2}\epsilon'_m L^m_{\nu-2}]. \quad .. \quad (8\ 1)$$

2nd order perturbation

$$\eta_2=\int_0^\infty(\rho^2-\eta_1)f_1 f_0 d\rho \Big/ \int_0^\infty f_0^2 d\rho$$

$$=\frac{\int_0^\infty (g(\rho))^2[\beta'_n L^m_{\nu+2}+\gamma'_n L^m_{\nu+1}+\delta'_n L^m_{\nu-1}+\epsilon'_n L^m_{\nu-2}]}{\int_0^\infty f_0^2 d\rho}\times$$

$$[-\tfrac{1}{2}\beta'_n L^m_{\nu+2}-\gamma'_n L^m_{\nu+1}+\delta'_n L^m_{\nu-1}+\tfrac{1}{2}\epsilon'_n L^m_{\nu-2}]d\rho$$

$$=-\tfrac{1}{2}\beta'^2_n[I]-\gamma'^2_n[II]+\delta'^2_n[III]+\tfrac{1}{2}\epsilon'^2_n[IV] \quad . \quad (8\ 2)$$

where $$[\mathrm{I}]=\int_0^\infty (g(\rho))^2(\mathrm{L}^m_{\nu+2}(\rho))^2 d\rho \Big/ \int_0^\infty (g(\rho))^2(\mathrm{L}^m_{\nu}(\rho))^2 d\rho$$

$$=\frac{[(\nu+2)!]^3/(\nu+2-m)!}{(\nu!)^3/(\nu-m)!}$$

$$=\frac{(n+m+2)^3(n+m+1)^3}{(n+2)(n+1)}$$

[See Condon and Morse, "Quantummechanics," sec 20, p 63, equa (20 5)]

And in a similar manner we can derive

$$[\mathrm{II}] =\int_0^\infty (g(\rho))^2 (\mathrm{L}^m_{\nu+1}(\rho))^2 d\rho \Big/ \int_0^\infty (g(\rho))^2(\mathrm{L}^m_{\nu}(\rho))^2 d\rho=(n+m+1)^3/(n+1) ,$$

$$[\mathrm{III}]=\int_0^\infty (g(\rho))^2(\mathrm{L}^m_{\nu-1}(\rho))^2 d\rho \Big/ \int_0^\infty (g(\rho))^2(\mathrm{L}^m_{\nu}(\rho))^2 d\rho=n/(n+m)^3 ,$$

$$[\mathrm{IV}]=\int_0^\infty (g(\rho))^2(\mathrm{L}^m_{\nu-2}(\rho))^2 d\rho \Big/ \int_0^\infty (g(\rho))^2(\mathrm{L}^m_{\nu}(\rho))^2 d\rho=n(n-1)/(n+m)^3$$

$$(n+m-1)^3$$

Hence substituting the values of these integrals (or rather ratios of integrals) and those of β'_n, γ'_n, in (8 2) we get finally

$$\eta_2=\tfrac{1}{2}[n(n-1)(n+m)(n+m-1)-(n+1)(n+2)(n+m+1)(n+m+2)]$$
$$+4[n(n+m)(2n+m)^2-(n+1)(n+m+1)(2n+m+2)^2] \quad .. \quad (8\ 3)$$

Again suppose

$$f_2=g(\rho)[\mathrm{E}'\mathrm{L}^m_{\nu+4}+\mathrm{D}'\mathrm{L}^m_{\nu+3}+\mathrm{C}'\mathrm{L}^m_{\nu+2}+\mathrm{B}'\mathrm{L}^m_{\nu+1}+\mathrm{A}_0\mathrm{L}^m_{\nu}+\mathrm{B}''\mathrm{L}^m_{\nu-1}$$
$$+\mathrm{C}''\mathrm{L}^m_{\nu-2}+\mathrm{D}''\mathrm{L}^m_{\nu-3}+\mathrm{E}''\mathrm{L}^m_{\nu-4}]$$

Now, as before, to calculate these coefficients E′, D′, .substitute in equa (7 3) and equate coefficients of $\mathrm{L}^m_{\nu+4}$,....we get

$$-4\mathrm{E}'=-\tfrac{1}{2}\beta'_n\beta'_{n+2}, \qquad -3\mathrm{D}'=-\tfrac{1}{2}\beta'_n\gamma'_{n+2}-\gamma'_n\beta'_{n+1};$$

$$-2C'=\tfrac{1}{2}\beta'_n(A_n-A_{n+2})-\gamma'_n\gamma'_{n+1}\,;$$

$$-\ B'=-\tfrac{1}{2}\beta'_n\delta'_{n+2}+\delta'_n\beta'_{n-1}+\gamma'_n(A_n-A_{n+1})\,;$$

$$A_0=-\tfrac{1}{2}\beta'_n\epsilon'_{n+2}-\gamma'_n\delta'_{n+1}+\delta'_n\gamma'_{n-1}+\tfrac{1}{2}\epsilon'_n\beta'_{n-2}-\eta_2=0,$$

from (8·2), since

$$\epsilon'_{n+2}=\beta'_n[\mathrm{I}],\ \delta'_{n+1}=\gamma'_n[\mathrm{II}],\ \gamma'_{n-1}=\delta'_n[\mathrm{III}],\ \beta'_{n-2}=\epsilon'_n[\mathrm{IV}],$$

$$B''=\tfrac{1}{2}\epsilon'_n\gamma'_{n-2}-\gamma'_n\epsilon'_{n+1}+\delta'_n(A_{n-1}-A_n),$$

$$2C''=\delta'_n\delta'_{n-1}+\tfrac{1}{2}\epsilon'_n(A_{n-2}-A_n),$$

$$3D''=\delta'_n\epsilon'_{n-1}+\tfrac{1}{2}\epsilon'_n\delta'_{n-2}, \qquad 4E''=\tfrac{1}{2}\epsilon'_n\epsilon'_{n-2}$$

3rd order perturbation

$$\eta_3=\left[\int_0^\infty(\rho^2-\eta_1)f_2f_0\,d\rho-\eta_2\int_0^\infty f_1f_0\,d\rho\right]\Big/\int_0^\infty f_0^2\,d\rho$$

$$=\left[\int_0^\infty f_2(\rho^2f_0)\,d\rho-\eta_1\int_0^\infty f_2f_0\,d\rho-\eta_2\int_0^\infty f_1f_0\,d\rho\right]\Big/\int_0^\infty f_0^2\,d\rho$$

$$=\Big[\int_0^\infty(g(\rho))^2\{E'L^m_{\nu+4}+\dots+E''L^m_{\nu-4}\}\{\beta'_nL^m_{\nu+2}+\dots+A_nL^m_\nu+\dots+\epsilon'_nL^m_{\nu-2}\}\,d\rho$$

$$-\eta_1\int_0^\infty(g(\rho))^2\,\{E'L^m_{\nu+4}+\dots+E''L^m_{\nu-4}\}\{L^m_\nu\}\,d\rho$$

$$-\eta_2\int_0^\infty(g(\rho))^2\{-\tfrac{1}{2}\beta'_nL^m_{\nu+2}-\gamma'_nL^m_{\nu+1}+\delta'_nL^m_{\nu-1}+\tfrac{1}{2}\epsilon'_nL^m_{\nu-2}\}$$

$$\times\{L^m_\nu\}\,d\rho\ \Big]\Big/\int_0^\infty f_0^2\,d\rho.$$

$$=-\tfrac{1}{2}\beta'_n(-2C')[\mathrm{I}]-\gamma'_n(-B')[\mathrm{II}]+\delta'_n(B'')[\mathrm{III}]+\tfrac{1}{2}\epsilon'_n(2C'')[\mathrm{IV}].$$

$$=-\tfrac{1}{2}\beta'_n\{\tfrac{1}{2}\beta'_n(A_n-A_{n+2})-\gamma'_n\gamma'_{n+1}\}[\mathrm{I}]$$

$$-\gamma'_n\{-\tfrac{1}{2}\beta'_n\delta'_{n+2}+\delta'_n\beta'_{n-1}+\gamma'_n(A_n-A_{n+1})\}[\mathrm{II}]$$

$$+\delta'_n\{\tfrac{1}{2}\epsilon'_n\gamma'_{n-2}\gamma'_n\epsilon'_{n+1}+\delta'_n(A_{n-1}-A_n)\}[\mathrm{III}]$$

$$+\tfrac{1}{2}\epsilon'_n\{\delta'_n\delta'_{n-1}+\tfrac{1}{2}\epsilon'_n(A_{n-2}-A_n)\}[IV]$$

$$=-\tfrac{1}{2}\{\tfrac{1}{2}\beta'_n(A_n-A_{n+2})-\gamma'_n\gamma'_{n+1}\}\{\epsilon'_{n+2}\}$$

$$-\{-\tfrac{1}{2}\beta'_n\delta'_{n+2}+\delta'\beta'_{n-1}+\gamma'_n(A_n-A_{n+1})\}\{\delta'_{n+1}\}$$

$$+\{\tfrac{1}{2}\epsilon'_n\gamma'_{n-2}-\gamma'_n\epsilon'_{n+1}+\delta'_n(A_{n-1}-A_n)\}\{\gamma'_{n-1}\}$$

$$+\tfrac{1}{2}\{\delta'_n\delta'_{n-1}+\tfrac{1}{2}\epsilon'_n(A_{n-2}-A_n)\}\{\beta'_{n-2}\}$$

$$=\tfrac{1}{4}[\beta'_n\epsilon'_{n+2}(A_{n+2}-A_n)+\beta'_{n-2}\epsilon'_n(A_{n-2}-A_n)]+\tfrac{1}{2}[\gamma'_n\gamma'_{n+1}\epsilon'_{n+2}$$

$$+\beta'_n\delta'_{n+1}\delta'_{n+2}+\epsilon'_n\gamma'_{n-1}\gamma'_{n-2}+\delta'_n\delta'_{n-1}\beta'_{n-2}]-\gamma'_n\delta'_{n+1}(A_n-A_{n+1})$$

$$+\delta'_n\gamma'_{n-1}(A_n-A_{n-1})-(\delta'_n\delta_{n+1}\beta'_{n-1}+\gamma'_n\gamma'_{n-1}\epsilon'_{n+1})$$

$$=4[(n+1)(n+2)(n+m+1)(n+m+2)(2n+m+2)(2n+m+4)$$

$$+n(n-1)(n+m)(n+m-1)(2n+m)(2n+m-2)]-8n(n+1)$$

$$(n+m)(n+m+1)(2n+m)(2n+m+2)+3(n+1)(n+2)(n+m+1)$$

$$(n+m+2)(2n+m+3)-3n(n-1)(n+m)(n+m-1)(2n+m-1)$$

$$+24(n+1)(n+m+1)(2n+m+2)^3-24n(n+m)(2n+m)^3 \quad .. \quad (9)$$

This can be expressed in terms of $\bigcirc(n)$, $\Phi(n)$, $\Psi(n)$, . as in equation (21) It is to be remembered that our present η's are co-efficients of the relevent λ''s with a change of sign, so that for all different perturbations it is implied that we shall get a negative sign to the expressions obtained in the body of the paper

Again, suppose

$$f_8=g(\rho)[G'_0L^m_{\nu+6}+F'_0L^m_{\nu+5}+E'_0L^m_{\nu+4}+D'_0L^m_{\nu+3}+C'_0L_{\nu+2}+B'_0L^m_{\nu+1}$$

$$+A_{0,0}L^m_\nu+B''_0L^m_{\nu-1}+C''_0L^m_{\nu-2}+D''_0L^m_{\nu-3}+E''_0L^m_{\nu-4}$$

$$+F''_0L^m_{\nu-5}+G''_0L^m_{\nu-6}] \qquad (9\,1)$$

And substitute the value of f_8 in equation (7 4) We get by equating the co-efficients of $L^m_{\nu+6}$,

$$-2C'_0 = E'\epsilon'_{n+4} + D'\delta'_{n+3} + C'A_{n+2} + B'\gamma'_{n+1} - \eta_2 B - \eta_1 C',$$
$$2C''_0 = B''\delta'_{n-1} + C''A_{n-2} + D''\gamma'_{n-3} + E''\beta_{n-4} - \eta_2 E - \eta_1 C'',$$
$$-B'_0 = D'\epsilon'_{n-1} + C'\delta'_{n+2} + B'A_{n+1} + B''\beta'_{n-1} - \eta_2 C - \eta_1 B',$$
$$B''_0 = B'\epsilon'_{n+1} + B''A_{n-1} + C''\gamma'_{n-2} + D''\beta'_{n-3} - \eta_2 D - \eta_1 B'',$$
$$A_{0,0} = C'\epsilon'_{n+2} + B'\delta'_{n+1} + B''\gamma'_{n-1} + C''\beta'_{n-2} - \eta_3 = 0,$$

on simplification

The final values of C'_0, C''_0, B'_0 and B''_0, obtained after simplification are written down below :

$$C'_0 = (-\tfrac{1}{2}\beta'_n)\{\tfrac{1}{8}\epsilon'_{n+4}\beta'_{n+2} + \tfrac{1}{6}\delta'_{n+3}\gamma'_{n+2} + \tfrac{1}{4}A_{n+2}(A_{n+2} - A_n)$$
$$+ \tfrac{1}{2}\gamma'_{n+1}\delta'_{n+2} + \tfrac{1}{2}\eta_2\} + (-\gamma'_n)\{\tfrac{1}{6}\beta'_{n+1}\delta'_{n+3} + \tfrac{1}{2}A_{n+2}\gamma'_{n+1}$$
$$+ \tfrac{1}{2}A_{n+1}\gamma'_{n+1} - \tfrac{1}{2}A_n\gamma'_{n+1}\} + \delta'_n(\tfrac{1}{2}\beta'_{n-1}\gamma'_{n+1}),$$

$$C''_0 = (\tfrac{1}{2}\epsilon'_n)\{\tfrac{1}{8}\beta'_{n-4}\epsilon'_{n-4} + \tfrac{1}{6}\gamma'_{n-3}\delta'_{n-2} + \tfrac{1}{4}A_{n-2}(A_{n-2} - A_n)$$
$$+ \tfrac{1}{2}\delta'_{n-1}\gamma'_{n-2} - \tfrac{1}{2}\eta_2\} + \delta'_n\{\tfrac{1}{6}\epsilon'_{n-1}\gamma'_{n-3} + \tfrac{1}{4}A_{n-2}\delta'_{n-1}$$
$$+ \tfrac{1}{2}\delta'_{n-1}(A_{n-1}A_n)\} + (-\tfrac{1}{2}\gamma'_n)(\delta'_{n-1}\epsilon'_{n+1}),$$

$$B'_0 = (-\tfrac{1}{2}\beta'_n)\{\tfrac{1}{3}\epsilon'_{n+3}\gamma'_{n+2} + \tfrac{1}{2}\delta'_{n+2}A_{n+2} + \delta'_{n+2}A_{n+1} - \tfrac{1}{2}\delta'_{n+2}A_n\}$$
$$+ (-\gamma_n)\{\tfrac{1}{3}\epsilon'_{n+3}\beta'_{n+1} + \tfrac{1}{2}\delta'_{n+2}\gamma'_{n+1} + A_{n+1}(A_{n+1} - A)$$
$$-\beta'_{n-1}\epsilon'_{n+1} + \eta_2\} + (\delta'_n)\{\beta'_{n-1}(A_{n+1} - A_{n-1} + A_n)\} - \tfrac{1}{2}\epsilon'_n\beta'_{n-1}\gamma'_{n-2}$$

$$B''_0 = (-\tfrac{1}{2}\beta'_n)(-\epsilon'_{n+n}\delta'_{n+2}) + (-\gamma'_n)\{\epsilon'_{n+1}(A_{n-1} - A_{n+1} + A_n)\}$$
$$+ (\delta'_n)\{-\epsilon'_{n+1}\beta'_{n-1} + A_{n-1}(A_{n-1} - A_n) + \tfrac{1}{2}\gamma'_{n-2}\delta'_{n-1}$$
$$+ \tfrac{1}{3}\beta'_{n-3}\epsilon'_{n-1} - \eta_2\} + (\tfrac{1}{2}\epsilon'_n)\{\gamma'_{n-2}(A_{n-1} - \tfrac{1}{2}A_n + \tfrac{1}{2}A_{n-2})$$
$$+ \tfrac{1}{3}\beta'_{n-3}\delta'_{n-2}\}$$

4th order perturbation :

$$\eta_4 = \left[\int_0^\infty \rho^2 f_3 f_0 d\rho - \eta_1 \int_0^\infty f_3 f_0 d\rho - \eta_2 \int_0^8 f_2 f_0 d\rho - \eta_3 \int_0^\infty f_1 f_0 d\rho\right] \Big/ \int_0^\infty f_0^2 d\rho$$
$$= C'_0\beta'_n[I] + B'_0\gamma'_n[II] + B''_0\delta'_n[III] + C''_0\epsilon'_n[IV] - \eta_2 A_0 - \eta_3 A - \eta_1 A_{0,0}$$
$$= A_{0,0}(A_n - \eta_1) - \eta_2 A_0 - \eta_3 A + C'_0\beta'_n[I] + B'_0\gamma'_n[II]$$
$$+ B''_0\delta'_n[III] + C''_0\epsilon'_n[IV]$$
$$= (\beta'_n C'_0)[I] + (\gamma'_n B'_0)[II] + (\delta'_n B''_0)[III] + (\epsilon'_n C''_0)[IV] \qquad \ldots \quad (10)$$

Whence substituting the values of C'_0, C''_0, B'_0, B''_0, and making simplifications we get :

$$
\begin{aligned}
\eta_4 = & \tfrac{1}{8}A_{n+2}[\beta'_n\epsilon'_{n+2}(A_n - A_{n+2})] + \tfrac{1}{8}A_{n-2}[\epsilon'_n\beta'_{n-2}(A_{n-2} - A_n)] \\
& + \tfrac{1}{4}[\beta'_n\delta'_{n+1}\delta'_{n+2}(A_n - A_{n+2}) + \gamma'_{n-1}\gamma'_{n-2}\epsilon'_n(A_{n-2} - A_n) \\
& - \gamma'_n\gamma'_{n+1}\epsilon'_{n+2}A'_{n+2} + \beta'_{n-2}\delta'_n\delta'_{n-1}A_{n-2}] \\
& + \tfrac{1}{2}[\gamma'_n\gamma'_{n+1}\epsilon'_{n+2}(A_nA_{n+1}) + \delta'_n\delta'_{n-1}\beta'_{n-2}(A_{n-1} - A_n) \\
& + \gamma'_{n-1}\gamma'_{n-2}\epsilon'_nA_{n-1} - \delta'_{n+1}\delta'_{n+2}\beta'_nA_{n+1} + \gamma'_{n-1}\delta'_nA_{n-1}(A_{n-1} - A_n) \\
& - \gamma'_n\delta'_{n+1}A_{n+1}(A_{n+1} - A_n)] + [\beta'_{n-1}\delta'_n\delta'_{n+1}(A_{n+1} + A_n - A_{n-1}) \\
& - \epsilon'_{n+1}\gamma'_n\gamma'_{n-1}(A_{n-1} + A_n - A_{n+1})] + [\beta'_{n-1}\gamma'_n\delta'_{n+1}\epsilon'_{n+1} \\
& - \beta'_{n-1}\gamma'_{n-1}\delta'_n\epsilon'_{n+1}] + \tfrac{1}{2}[(\beta'_{n-1}\gamma'_{n+1}\delta'_n\epsilon'_{n+2} - \beta'_{n-2}\gamma'_n\delta'_{n-1}\epsilon'_{n+1}) \\
& + (\beta'_n\gamma'_{n-1}\delta'_{n+2}\epsilon'_{n+1} - \beta'_{n-1}\gamma'_{n-2}\delta'_{n+1}\epsilon'_n) + (\gamma'_{n-1}\gamma'_{n-2}\delta'_n\delta'_{n-1} \\
& - \gamma'_n\gamma'_{n+1}\delta'_{n+1}\delta'_{n+2})] + \tfrac{1}{3}[\beta'_{n-3}\gamma'_{n-1}\delta'_n\epsilon'_{n-1} - \beta'_{n+1}\gamma'_n\delta'_{n+1}\epsilon'_{n+3}] \\
& + \tfrac{1}{4}[\beta'_{n-2}\gamma'_{n-2}\delta'_{n-1}\epsilon'_n - \beta'_n\gamma'_{n+1}\delta'_{n+2}\epsilon'_{n+2}] + \tfrac{1}{6}[\beta'_{n-2}\gamma'_{n-3}\delta'_n\epsilon'_{n-1} \\
& - \beta'_{n+1}\gamma'_n\delta'_{n+3}\epsilon'_{n+2}] + \tfrac{1}{6}[\beta'_{n-3}\gamma'_{n-1}\delta'_{n-2}\epsilon'_n - \beta'_n\gamma'_{n+2}\delta'_{n+1}\epsilon'_{n+3}] \\
& + \tfrac{1}{12}[\beta'_{n-2}\gamma'_{n-3}\delta'_{n-2}\epsilon'_n - \beta'_n\gamma'_{n+2}\delta'_{n+3}\epsilon'_{n+2}] + \tfrac{1}{16}[\beta'_{n-2}\beta'_{n-4}\epsilon'_n\epsilon'_{n-2} \\
& - \beta'_n\beta'_{n+2}\epsilon'_{n+2}\epsilon'_{n+4}] + \eta_2[-\tfrac{1}{4}\beta'_n\epsilon'_{n+2} - \tfrac{1}{4}\epsilon'_n\beta'_{n-2} - \gamma'_n\delta'_{n+1} - \gamma'_{n-1}\delta'_n]
\end{aligned}
$$

where

$$\eta_2 = -\tfrac{1}{2}\beta'_n\epsilon'_{n+2} - \gamma'_n\delta'_{n+1} + \gamma'_{n-1}\delta'_n + \tfrac{1}{2}\epsilon'_n\beta'_{n-2}$$

APPENDIX II

This note will serve as a key to our arriving at the results given in sec. 8.

$\underline{\theta(r) \equiv (r+1)(r+2)(r+m+1)(r+m+2)\,;}$

$$\theta(r) = r^4 + (2m+6)r^3 + (m^2+9m+13)r^2 + (3m^2+13m+12)r + 2(m^2+3m+2)\,;$$

$$\theta(r+1) = r^4 + (2m+10)r^3 + (m^2+15m+37)r^2 + (5m^2+37m+60)r + (6m^2+30m+36)\,,$$

$$\theta(r+2)=r^4+(2m+14)r^3+(m^2+21m+73)r^2+(7m^2+73m$$
$$+168)r+(12m^2+84m+144)\ ;$$

$$\theta(r-1)=r^4+(2m+2)r^3+(m^2+3m+1)r^2+(m^2+m)r\ ;$$

$$\theta(r-2)=r^4+(2m-2)r^3+(m^2-3m+1)r^2+(-m^2+m)r\ ,$$

$$\theta(r-3)=r^4+(2m-6)r^3+(m^2-9m+13)r^2+(-3m^2+13m$$
$$-12)r+2(m^2-3m+2)\ ;$$

$$\theta(r-4)=r^4+(2m-10)r^3+(m^2-15m+37)r^2+(-5m^2$$
$$+37m-60)r+(6m^2-30m+36)\ ;$$

$\underline{\psi(r)\equiv 4(2r+m+2)\Delta(r+1)\ ;\ \Delta(r)\equiv r(r+m)(2r+m)}$

$$\psi(r)=4[4r^4+(8m+16)r^3+(5m^2+24m+24)r^2+(m^3+10m^2$$
$$+24m+16)r+(m^3+5m^2+8m+4)]\ ,$$

$$\psi(r+1)=4[4r^4+(8m+32)r^3+(5m^2+48m+96)r^2+(m^3+20m^2$$
$$+96m+128)r+(2m^3+20m^2+64m+64)]\ ,$$

$$\psi(r+2)=4[4r^4+(8m+48)r^3+(5m^2+72m+216)r^2+(m^3+30m^2$$
$$+216m+432)r+(3m^3+45m^2+216m+324)]\ ;$$

$$\psi(r-1)=4[4r^4+8mr^3+5m^2r^2+m^3r]\ ,$$

$$\psi(r-2)=4[4r^4+(8m-16)r^3+(5m^2-24m+24)r^2+(m^3-10m^2$$
$$+24m-16)r+(-m^3+5m^2-8m+4)]\ ;$$

$$\psi(r-3)=4[4r^4+(8m-32)r^3+(5m^2-48m+96)r^2+(m^3-20m^2$$
$$+96m-128)r+(-2m^3+20m^2-64m+64)]\ ;$$

$\underline{F(r)\equiv r(r+m)\Delta(r-1)\Delta(r+1)}$

$$F(r)=4r^8+16mr^7+(25m^2-12)r^6+(19m^3-36m)r^5+(7m^4-42m^2$$
$$+12)r^4+m(m^4-24m^2+24)r^3-(7m^4-17m^2+4)r^2-$$
$$(m^5-5m^3+4m)r\ ;$$

$$F(r+1)=4r^8+(16m+32)r^7+(25m^2+112m+100)r^6+(19m^3+150m^2+300m+152)r^5+(7m^4+95m^3+333m^2+380m+112)r^4+(m^5+28m^4+166m^3+332m^2+224m+32)r^3+(3m^5+35m^4+118m^3+140m^2+48m)r^2+(2m^5+14m^4+28m^3+16m^2),$$

$$F(r-1)=4r^8+(16m-32)r^7+(25m^2-112m+100)r^6+(19m^3-150m^2+300m-152)r^5+(7m^4-95m^3+333m^2-380m+112)r^4+(m^5-28m^4+166m^3-332m^2+224m-32)r^3+(-3m^5+35m^4-118m^3+140m^2-48m)r^2+(2m^5-14m^4+28m^3-16m^2),$$

$$F(r+2)=4r^8+(16m+64)r^7+(25m^2+224m+436)r^6+(19m^3+300m^2+1308m+1648)r^5+(7m^4+190m^3+1458m^2+4120m+3772)r^4+(m^5+56m^4+736m^3+3664m^2+7544m+5344)r^3+(6m^5+161m^4+1376m^3+5009m^2+8016m+4572)r^2+(11m^5+196m^4+1237m^3+3524m^2+4572m+2160)r+(6m^5+84m^4+426m^3+996m^2+1080m+432)$$

$$\{I\}=\tfrac{1}{3}[\{F(r+2)-F(r-1)\}-3\{F(r+1)-F(r)\}]$$

$$=\tfrac{1}{3}[1344r^5+3360(m+1)r^4+(3000m^2+6720m+5280)r^3+(1140m^3+4500m^2+7920m+4560)r^2+(168m^4+1140m^3+3492m^2+4560m+2160)r+(6m^5+84m^4+426m^3+996m^2+1080m+432)],$$

$$=4[448r^5+1120(m+1)r^4+(1000m^2+2240m+1760)r^3+(380m^3+1500m^2+2640m+1520)r^2+(56m^4+380m^3+1164m^2+1520m+720)r+(2m^5+28m^4+142m^3+332m^2+360m+144)]$$

$$\psi(r)\psi(r+1)=16[16r^8+(64m+192)r^7+(104m^2+672m+992)r^6+(88m^3+936m^2+2976m+2880)r^5+(41m^4+660m^3+3452m^2+7200m+5136)r^4+(10m^5+246m^4+1944m^3+6672m^2+10272m+5760)r^3+(m^6+45m^5+541m^4+2808m^3+7124m^2+8640m+3968)r^2$$

$$+(3m^6+65m^5+516m^4+1988m^3+3984m^2+3968m+1536)r$$
$$+(2m^6+30m^5+180m^4+552m^3+912m^2+768m+256)]\,,$$

$$\psi(r-1)\psi(r-2)=16[16r^8+(64m-64)r^7+(104m^2-224m+96)r^6$$
$$+(88m^3-312m^2+288m-64)r^5+(41m^4-220m^3+332m^2$$
$$-160m+16)r^4+(10m^5-82m^4+184m^3-144m^2+32m)r^3$$
$$+(m^6-15m^5+49m^4-56m^3+20m^2)r^2+(-m^6+5m^5-8m^4$$
$$+4m^3)r]\,;$$

$$\psi(r)+\psi(r-1)=4[8r^4+(16m+16)r^3+(10m^2+24m+24)r^2+(2m^3$$
$$+10m^2+24m++16)r+(m^4+5m^3+8m+4)]\,;$$

$$\psi(r)-\psi(r-1)=4[16r^3+(24m+24)r^2+(10m^2+24m+16)r+(m^3$$
$$+5m^2+8m+4)]\,,$$

$$[\psi(r)+\psi(r-1)][\psi(r)-\psi(r-1)]=16[128r^7+448(m+1)r^6+(624m^2$$
$$+1344m+896)r^5+(440m^3+1560m^2+2240m+1120)r^4+(164m^4$$
$$+880m^3+2080m^2+2240m+896)r^3+(30m^5+246m^4+880m^3$$
$$+1560m^2+1844m+448)r^2+(2m^6+30m^5+164m^4+440m^3$$
$$+624m^2+448m+128)r+(m^6+10m^5+41m^4+88m^3+104m^2$$
$$+64m+16)]\,;$$

$$\{II\}=\tfrac{1}{2}[\psi(r)\psi(r+1)-\psi(r-1)\psi(r-2)]-[\psi(r)+\psi(r+1)][\psi(r)-\psi(r-1)]$$
$$=8[1152r^5+2880(m+1)r^4+(2656m^2+5760m+3968)r^3+(1104m^3$$
$$+3984m^2+5952m+3072)r^2+(196m^4+1104m^3+2736m^2$$
$$+3072m+1280)r+(10m^5+98m^4+376m^3+704m^2$$
$$+640m+224)].$$

$$\theta(r)\theta(r+2)=r^8+(4m+20)r^7+(6m^2+70m+170)r^6+(4m^3+90m^2$$
$$+510m+800)r^5+(m^4+50m^3+545m^2+2000m+2273)r^4+(10m^4$$
$$+240m^3+1700m^2+4546m+3980)r^3+(35m^4+550m^3+2869m^2$$

$$+5970m+4180)r^2+(50m^4+596m^3+2470m^2+4180m+2400)r$$
$$+(24m^4+240m^3+840m^2+1200m+576);$$

$$\theta(r-2)\theta(r-4)=r^8+(4m-12)r^7+(6m^2-42m+58)r^6+(4m^3-54m^2$$
$$+174m-144)r^5+(m^4-30m^3+185m^2-360m+193)r^4+(-6m^4$$
$$+80m^3-300m^2+386m-132)r^3+(11m^4-90m^3+229m^2-198m$$
$$+36)r^2+(-6m^4+36m^3-66m^2+36m)r;$$

$$\theta(r)\theta(r+2)-\theta(r-2)\theta(r-4)=32r^7+112(m+1)r^6+(144m^2+336m$$
$$+944)r^5+(80m^3+360m^2+2360m+2080)r^4+(16m^4+160m^3$$
$$+2000m^2+4160m+4112)r^3+(24m^4+640m^3+2640m^2+6168m$$
$$+4144)r^2+(56m^4+560m^3+2536m^2+4144m+2400)r+(24m^4$$
$$+240m^3+840m^2+1200m+576);$$

$$\theta^2(r)-\theta^2(r-2)=[\theta(r)+\theta(r-2)][\theta(r)-\theta(r-2)]=16r^7+56(m+1)r^6$$
$$+(72m^2+168m+184)r^5+(40m^3+180m^2+460m+320)r^4+(8m^4$$
$$+80m^3+392m^2+640m+360)r^3+(12m^4+128m^3+408m^2+540m$$
$$+248)r^2+(12m^4+88m^3+228m^2+248m+96)r+(4m^4+24m^3$$
$$+52m^2+48m+16);$$

$$\{\text{III}\}=\tfrac{1}{16}[\theta(r)\theta(r+2)-2\{\theta^2(r)-\theta^2(r-2)\}-\theta(r-2)\theta(r-4)]$$

$$=\tfrac{1}{16}[576r^5+1440(m+1)r^4+(1216m^2+2880m+3392)r^3+(384m^3$$
$$+1824m^2+5088m+3648)r^2+(32m^4+384m^3+2080m^2$$
$$+3648m+2208)r+(16m^4+192m^3+736m^2+1104m+544)];$$

$$=36r^5+90(m+1)r^4+(76m^2+180m+212)r^3+(24m^3+114m^2$$
$$+318m+228)r^2+(2m^4+24m^3+130m^2+228m+138)r+(m^4$$
$$+12m^3+46m^2+69m+34).$$

$$\begin{aligned}\theta(r)\psi(r+2)=4[&4r^8+(16m+72)r^7+(25m^2+252m+556)r^6+(19m^3\\&+336m^2+1668m+2400)r^5+(7m^4+210m^3+1842m^2+6000m\\&+6316)r^4+(m^5+60m^4+904m^3+5256m^2+12632m+10344)r^3\\&+(6m^5+185m^4+1884m^3+8201m^2+15516m+10260)r^2+(11m^5\\&+240m^4+1885m^3+6600m^2+10260m+5616)r+(6m^5+108m^4\\&+714m^3+2124m^2+2808m+1296)]\,,\end{aligned}$$

$$\begin{aligned}\theta(r-2)\psi(r-3)=4[&4r^8+(16m-40)r^7+(25m^2-140m+164)r^6\\&+(19m^3-186m^2+492m-352)r^5+(7m^4-115m^3+537m^2\\&-880m+416)r^4+(m^5-32m^4+254m^3-748m^2+832m-256)r^3\\&+(-3m^5+47m^4-242m^3+500m^2-384m+64)r^2+(2m^5-22m^4\\&+84m^3-128m^2+64m)r]\,;\end{aligned}$$

$$\begin{aligned}\tfrac{1}{12}[\theta(r)\psi(r+2)-\theta(r-2)\psi(r-3)]=\tfrac{1}{3}[&112r^7+392(m+1)r^6+(522m^2\\&+1176m+2752)r^5+(325m^3+1305m^2+6880m+5900)r^4\\&+(92m^4+650m^3+6004m^2+11800m+10600)r^3+(9m^5+138m^4\\&+2126m^3+7701m^2+15900m+10196)r^2+(9m^5+262m^4\\&+1801m^3+6728m^2+10196m+5616)r+(6m^5+108m^4+714m^3\\&+2124m^2+2808m+1296)]\,;\end{aligned}$$

$$\begin{aligned}\theta(r)\psi(r+1)=4[&4r^8+(16m+56)r^7+(25m^2+196m+340)r^6+(19m^3\\&+262m^2+1020m+1168)r^5+(7m^4+165m^3+1133m^2+2920m\\&+2480)r^4+(m^5+48m^4+566m^3+2584m^2+4960m+3328)r^3+\\&(5m^5+121m^4+956m^3+3272m^2+4992m+2752)r^2+(8m^5\\&+132m^4+792m^3+2176m^2+2752m+1280)r+(4m^5+52m^4\\&+256m^3+592m^2+640m+256)]\,;\end{aligned}$$

$$\theta(r-2)\psi(r-2)=4[4r^8+(16m-24)r^7+(25m^2-84m+60)r^6+(19m^3$$
$$-112m^2+180m-80)r^5+(7m^4-70m^3+198m^2-200m+56)r^4$$
$$+(m^5-20m^4+96m^3-172m^2+120m-24)r^3+(-2m^5+19m^4$$
$$-58m^3+73m^2-36m+4)r^2+(m^5-6m^4+13m^3-12m^2+4m)];$$

$$\tfrac{1}{4}[\theta(r)\psi(r+1)-\theta(r-2)\psi(r-2)]=80r^7+280(m+1)r^6+(374m^2$$
$$+840m+1248)r^5+(235m^3+935m^2+3120m+2420)r^4+(68m^4$$
$$+470m^3+2756m^2+4840m+3352)r^3+(7m^5+102m^4+1014m^3$$
$$+3199m^2+5028m+2748)r^2+(7m^5+138m^4+779m^3+2188m^2$$
$$+2748m\ +2748m+1280)r+(4m^5+52m^4+256m^3+592m^2$$
$$+640m+256)\ ,$$

$$\psi(r)\theta(r+1)=4[4r^8+(16m+56)r^7+(25m^2+196m+332)r^6(19m^3$$
$$+264m^2+996m+1088)r^5+(7m^4+170m^3+1122m^2+2720m$$
$$2156)r^4+(m^5+52m^4+584m^3+2456m^2+4312m+2648)r^3+(6m^5$$
$$+137m^4+964m^3+2921m^2+3972m+1972)r^2+(11m^5+152m^4$$
$$+765m^3+1792m^2+1972m+816)r+(6m^5+60m^4+234m^3$$
$$+444m^2+408m+144)]\ ,$$

$$\psi(r-1)\theta(r-3)=4[4r^8+(16m-24)r^7+(25m^2-84m+52)r^6+(19m^3$$
$$-114m^2+156m-48)r^5+(7m^4-75m^3+177m^2-120m+16)r^4$$
$$+(m^5-24m^4+94m^3-108m^2+32m)r^3+(-3\ m^5+23m^4+42m^3$$
$$+20m^2)r^2+(2m^5-6m^4+4m^3)r]\ ;$$

$$\tfrac{1}{4}[\psi(r)\theta(r+1)-\psi(r-1)\theta(r-3)]=\tfrac{1}{4}[80r^7+280(m+1)r^6+(378m^2$$
$$+840m+1136)r^5+(245m^3+945m^2+2640m+2140)r^4+(76m^4$$
$$490m^3+2564m^2+4280m+2648)r^3+(9m^5+114m^4+1006m^3$$
$$+2901m^2+3972m+1972)r^2+(9m^5+158m^4+761m^3+1792m^2$$
$$+1972m+816)r+(6m^5+60m^4+234m^3+444m^2+408m+144)]$$

$$[\psi(r)-\psi(r-1)][\theta(r-1)]=4[16r^7+56(m+1)r^6+(74m^2+168m+80)r^5$$
$$+(45m^3+185m^2+200m+60)r^4+(12m^4+90m^3+172m^2+120m$$
$$+24)r^3+(m^5+18m^4+58m^3+73m^2+36m+4)r^2+(m^5+6m^4$$
$$+13m^3+12m^2+4m)r]\,,$$

$$\tfrac{1}{2}[\theta(r)-\theta(r-2)][\psi(r)+\psi(r-1)]=2[64r^7+224(m+1)r^6+(304m^2$$
$$+672m+480)r^5+(200m^3+760m^2+1200m+640)r^4+(64m^4$$
$$+400m^3+1080m^2+1280m+576)r^3+(8m^5+96m^4+420m^3$$
$$+860m^2+864m+336)r^2+(8m^5+64m^4+220m^3+388m^2+336m$$
$$+112)r+(2m^5+16m^4+50m^3+76m^2+56m+16)]\,,$$

$$\tfrac{1}{4}[\theta(r)+\theta(r-2)][\psi(r)-\psi(r-1)]=32r^7+112(m+1)r^6+(148m^2+336m$$
$$+352)r^5+(90m^3+370m^2+880m+600)r^4+(24m^4+180m^3+784m^2$$
$$+1200m+592)r^3+(2m^5+36m^4+296m^3+806m^2+888m+344)r^2$$
$$+(2m^5+44m^4+206m^3+396m^2+344m+112)r+(2m^5+16m^4$$
$$+50m^3+76m^2+56m+16)\,,$$

$$\{IV\}=\tfrac{1}{12}[\theta(r)\psi(r+2)-\theta(r-2)\psi(r-3)]+\tfrac{1}{4}[\theta(r)\psi(r+1)-\theta(r-2)$$
$$\psi(r-2)]+\tfrac{1}{3}[\psi(r)\theta(r+1)-\psi(r-1)\theta(r-3)]-[\psi(r)-\psi(r-1)]$$
$$[\theta(r-1)]+\tfrac{1}{2}[\theta(r)-\theta(r-2)][\psi(r)+\psi(r-1)]-\tfrac{1}{4}[\theta(r)+\theta(r-2)]$$
$$[\psi(r)-\psi(r-1)].$$

$$=2048r^5+5120(m+1)r^4+(4544m^2+10240m+8576)r^3+(1696m^3$$
$$+6816m^2+12864m+7744)r^2+(240m^4+1696m^3+5600m^2$$
$$+7744m+3904)r+(8m^5+120m^4+656m^3+1664m^2$$
$$+1952m+832).$$

APPENDIX III

$$A_{r+1}-A_r=6(2r+m+2)\ ; \quad A_{r-1}-A_r=-6(2r+m)\ ;$$

$$A_{r+2}-A_r=12(2r+m+3)\ , \quad A_{r-2}-A_r=-12(2r+m-1),$$

$$\begin{aligned}\delta'_{r+1}\gamma'_r(A_{r+1}-A_r)^2&=144\psi(r)\,(2r+m+2)^2=144[16r^6+48(m+2)r^5\\&+(56m^2+240m+240)r^4+(32m^3+224m^2+480m+320)r^3+(9m^4\\&+96m^3+336m^2+480m+240)r^2+(m^5+18m^4+96m^3+224m^2\\&+240m+96)r+(m^5+9m^4+32m^3+56m^2+48m+16)]\ ,\end{aligned}$$

$$\begin{aligned}\delta'_r\gamma'_{r-1}(A_{r-1}-A_r)^2&=144\psi(r-1)\,(2r+m)^2=144[16r^6+48mr^5\\&+56m^2r^4+32m^3r^3+9m^4r^2+m^5r]\ ,\end{aligned}$$

$$\begin{aligned}\{V\}&=\delta'_{r+1}\gamma'_r(A_{r+1}-A_r)^2-\delta'_r\gamma'_{r-1}(A_{r-1}-A_r)^2=144[96r^5\\&+240(m+1)r^4+(224m^2+480m+320)r^3+(96m^3+336m^2+480m\\&+240)r^2+(18m^4+96m^3+224m^2+240m+96)r+(m^5+9m^4\\&+32m^3+56m^2+48m+16)]\ ,\end{aligned}$$

$$\begin{aligned}\tfrac{1}{8}\epsilon'_{r+2}\beta'_r(A_{r+2}-A_r)^2&=18\theta(r)\,(2r+m+3)^2=18[4r^6+12(m+3)r^5\\&+(13m^2+90m+133)r^4+(6m^3+78m^2+266m+258)r^3+(m^4\\&+27m^3+172m^2+387m+277)r^2+(3m^4+39m^3+165m^2+277m\\&+156)r+(2m^4+18m^3+58m^2+78m+36)]\ ,\end{aligned}$$

$$\begin{aligned}\tfrac{1}{8}\epsilon'_r\beta'_{r-2}(A_{r-2}-A_r)^2&=18\theta(r-2)\,(2r+m-1)^2=18[4r^6+12(m-1)r^5\\&+(13m^2-30m+13)r^4+(6m^3-26m^2+26m-6)r^3+(m^4-9m^3\\&+16m^2-9m+1)r^2+(-m^4+3m^3-3m^2+m)r)]\ ,\end{aligned}$$

$$\begin{aligned}\{VI\}&=\tfrac{1}{8}[\epsilon'_{r+2}\beta'_r(A_{r+2}-A_r)^2-\epsilon'_r\beta'_{r-2}(A_{r-2}-A_r)^2]=18[48r^5\\&+120(m+1)r^4+(104m^2+240m+264)r^3+(36m^3+156m^2+396m\\&+276)r^2+(4m^4+36m^3+168m^2+276m+156)r+(2m^4+18m^3\\&+58m^2+78m+36)]\end{aligned}$$

$$\begin{aligned}(2r+m)(2r+m+1)(2r+m+2)&=8r^3+12(m+1)r^2+2(3m^2+6m+2)r\\&+m(m^2+3m+2)\ ;\end{aligned}$$

$$(2r+m)(2r+m-1)(2r+m-2)=8r^3+12(m-1)r^2+2(3m^2-6m+2)r$$
$$+m(m^2-3m+2)\,,$$

$$(2r+m+2)(2r+m+3)(2r+m+4)=8r^3+12(m+3)r^2+2(3m^2+18m$$
$$+26)r+(m^3+9m^2+26m+24)\,,$$

$$(2r+m+2)^2(2r+m+4)=8r^3+4(3m+8)r^2+2(3m^2+16m+20)r$$
$$+(m^3+8m^2+20m+16)\,,$$

$$(2r+m)^2(2r+m-2)=8r^3+4(3m-2)r^2+2(3m^2-4m)+(m^3-2m^2)\,,$$

$$\{\mathrm{VII}\}=24[W_r+W_{r-2}-4W_{r-1}],$$

where

$$W_r\equiv\theta(r)[(2r+m+2)(2r+m+3)(2r+m+4)(2r+m+2)^2(2r+m+4)]\,,$$

$$W_{r-2}\equiv\theta(r-2)[(2r+m)(2r+m-1)(2r+m-2)(2r+m)^2(2r+m-2)]\,,$$

$$W_{r-1}\equiv\theta(r-1)[(2r+m)(2r+m+1)(2r+m+2)]\,,$$

$$W_r=16r^7+(56m+164)r^6+(76m^2+492m+708)r^5+(50m^3+557m^2$$
$$+1770m+1668)r^4+(16m^4+294m^3+1602m^2+3336m+2316)r^3$$
$$+(2m^5+71m^4+633m^3+2259m^2+3474m+1896)r^2+(6m^5$$
$$+101m^4+591m^3+1562m^2+1896m+848)r\,,$$

$$W_{r-2}=16r^7+(56m-52)r^6+(76m^2-156m+60)r^5+(50m^3-177m^2$$
$$+150m-28)r^4+(16m^4-94m^3+134m^2-56m+4)r^3+(2m^5$$
$$-23m^4+51m^3-35m^2+6m)r^2+(-2m^5+7m^4-7m^3+2m^2)r\,,$$

$$W_{r-1}=8r^7+28(m+1)r^6+(38m^2+84m+36)r^5+(25m^3+95m^2$$
$$+90m+20)r^4+(8m^4+50m^3+80m^2+40m+4)r^3+(m^5+12m^4$$
$$+30m^3+25m^2+6m)r^2+(m^5+4m^4+5m^3+2m^2)r\,,$$

$$\{\mathrm{VII}\}=24[624r^5+1560(m+1)r^4+(1416m^2+3120m+2304)r^3$$
$$+(564m^3+2124m^2+3456m+1896)r^2+(92m^4+564m^3+1556m^2$$
$$+1896m+848)r+(4m^5+46m^4+202m^3+424m^2+424m+160)]$$

APPENDIX IV

$$\tfrac{1}{4}\,T(n,m) \equiv 21378n^4 - 53445(m+1)n^3 + (48990m^2 + 106890m$$
$$+76850)n^2 - (20040m^3 + 73485m^2 + 115275m + 61830\ n$$
$$+(3453m^4 + 20040m^3 + 52449m^2 + 61830m + 27073)$$
$$-106890n^2n_1n_2 + 106890(m+1)nn_1n_2 + 106890n_1^2n_2^2 - (40080m^2$$
$$+106890m + 123660)n_1n_2 + \quad . \quad . \quad \text{(1st part)}$$

$$1296[34n^2 - 51(m+1)n - 102n_1n_2 + 21m^2 + 51m + 35][n^2 - 2(m+1)n$$
$$+(m+1)^2 - 4n_1n_2] - 162[n^2 - 2(m+1)n + (m+1)^2 - 4n_1n_2]$$
$$[250n^2 - 250(m+1)n - 500n_1n_2 + 82m^2 + 250m + 278]$$
$$-9[34n^2 - 51(m+1)n - 102n_1n_2 + 21m^2 + 51m + 35]^2 - 13608[n^4$$
$$-4(m+1)n^3 + 6\ m+1)^2n^2 - 4(m+1)^3n + (m+1)^4 - 8n^2n_1n_2$$
$$+16(m+1)nn_1n_2 - 8(m+1)^2n_1n_2 + 16n_1^2n_2^2] \qquad \text{(2nd part)}$$
$$= \{\text{1st part}\} + 1296[34n^4 - 119(m+1)n^3 + (157m^2 + 323m + 171)n^2$$
$$-(93m^3 + 297m^2 + 325m + 121)n + (m+1)^2(21m^2 + 51m + 35)$$
$$+408n_1^2n_2^2 - 238n^2n_1n_2 + 408(m+1)nn_1n_2 - (186m^2 + 408m$$
$$+242)n_1n_2 - 162[250n^4 - 750(m+1)n^3 + (832m^2 + 1750m$$
$$+1028)n^2 - (m+1)(414m^2 + 1000m + 806)n + (m+1)^2(82m^2$$
$$+250m + 278)] - 1500n^2n_1n_2 + 2000(m+1)nn_1n_2 - (828m^2$$
$$+2000m + 1612)n_1n_2 + 2000n_1^2n_2^2 - 9[1156n^4 - 3468(m+1)n^3$$
$$+(4029m^2 + 8670m + 4981)n^2 - (2142m^3 + 7344m^2 + 8772m$$
$$+3570)n + (441m^4 + 2142m^3 + 2601m^2 + 3570m + 1125)$$
$$+10404n_1^2n_2^2 - 6936n^2n_1n_2 + 10404(m+1)nn_1n_2 - (4284m^2$$
$$+10404m + 7140)n_1n_2 - 13608[n^4 - 4(m+1)n^3 + 6(m+1)^2n^2$$
$$-4\ m+1)^3n + (m+1)^4 - 8n^2n_1n_2 + 16(m+1)nn_1n_2$$
$$-8(m+1)^2n_1n_2 + 16n_1^2n_2^2]$$

Collecting coefficients of n^4, n^3 .. we obtain :

$$\tfrac{1}{4}\,\mathrm{T}(n,m)=n^4\{21378+1296\cdot 34-162\cdot 250-9\cdot 1156-13608\}$$

$$-n^3(m+1)\ \{53445+119\cdot 1296-162\cdot 750-9\cdot 3468-4\cdot 13608\}$$

$$+n^2\{48990m^2+106890m+76850+1296(157m^2+323m$$

$$+171)-162(832m^2+1750m+1028)-9(4029m^2+8670m+4981)$$

$$-13608\cdot 6(m^2+2m+1)\}-n\{20040m^3+73485m^2+115275m$$

$$+61830+1296(93m^3+297m^2+325m+121)-162(414m^3$$

$$+1414m^2+1806m+806)-9(2142m^3+7344m^2+8772m$$

$$+3570)-4\cdot 13608(m^3+3m^2+3m+1)\}+\{(3453m^4+20040m^3$$

$$+52449m^2+61830m+27073)+1296(m+1)^2(21m^2+51m+35)$$

$$-162(m+1)^2(82m^2+250m+278)-9(441m^4+2142m^3$$

$$+2601m^2+3570m+1125)-13608(m+1)^4\}+n_1^2n_2^2\{106890$$

$$+1296\cdot 408-162\cdot 2000-9\cdot 10404-16\cdot 13608\}+n^2n_1n_2\{-106890$$

$$-1296\cdot 238+162\cdot 1500+9\cdot 6936+8\cdot 13608\}+nn_1n_2(m+1)$$

$$\{106890+1296\cdot 408-162\cdot 2000-9\cdot 10404-16\cdot 13608\}$$

$$-n_1n_2\{(40080m^2+106890m+123660)+1296(186m^2+408m$$

$$+242)-162(828m^2+2000m+1612)-9(4284m^2+10404m$$

$$+7140)-8\cdot 13608(m+1)^2\}.$$

$$=930n^4-525(m+1)n^3+(-231m^2+672m+5453)n^2-(-210m^3$$

$$-63m^2+1659m+1512)n+(-192m^4-210m^3+12840m^2$$

$$+1512m+3664)+294n_1^2n_2^2-1050n^2n_1n_2+294(m+1)nn_1n_2$$

$$-(-420m^2+294m+8024)n_1n_2.$$

APPENDIX V*

Notation : $(n_1^f, n_2^f, m^f+1) \longleftarrow (n_1^i, n_2^i, m^i+1)$, $(d=d_0\times 10^{24})$

(111)⟵(011)

$$d_0=D\{930(3^{14}-2^{14})-525(3^{13}-2^{13})+5453(3^{12}-2^{12})$$
$$-1512(3^{11}-2^{11})+3664(3^{10}-2^{10})+294\,3^{10}-1050\,3^{12}+294\,3^{11}$$
$$-3024\,3^{10}\}=D\{98062\,3^{10}-33132\,2^{10}\}=D\,10^9(5{,}79047$$
$$-0{,}0039272)=2{,}5746\,10^{-27}\,5{,}78655=\underline{1{,}485\,10^{-26}}$$

(102)⟵(002) :

$$d_0=D\{930(3^{14}-2^{14})-525.2\,3^{13}-2^{13})+5894(3^{12}-2^{12})$$
$$-2898(3^{11}-2^{11})+17614(3^{10}-2^{10})\}=D\{108956\,3^{10}$$
$$-41874\,2^{10}\}=D\,10^9(6{,}43316-0{,}042877)=2{,}5746\,10^{-27}$$
$$6{,}39029=\underline{1{,}642\,10^{-26}}$$

(201)⟵(101)

$$d_0=D\{930(3^{14}-2^{14})-525(3^{13}-2^{13})+5453(3^{12}-2^{12})$$
$$-1512(3^{11}-2^{11})+3664\,3^{10}-2^{10})\}$$
$$D\{109360\,3^{10}-33132\,2^{10}\}=D\,10^9(6{,}4576-0{,}0393)$$
$$=2{,}5746.10^{-27}\,6{,}4183 \qquad =\underline{1{,}649\,10^{-26}}.$$

(003)⟵(002) ·

$$d_0=D\{930(3^{14}-2^{14})-525(3\,3^{13}-2\,2^{13})+(-231\,2^2$$
$$+672\,2+5453)3^{12}-(-231\,1^2+672.1+5453)2^{12}-(-210\,2^3$$
$$-63\,2^2+1659\,2+1512)3^{11}-(-210\,1^3-63\,1^2+1649$$
$$+1512)2^{11}+(-192\,2^4-210\,2^3+12840\,2^2+1512.2+3664)3^{10}$$
$$-(-192\,1^4-210\,1^3+12840\,1^2+1512\,1+3664)2^{10}\}$$
$$=D\{133288\,3^{10}-41874\,2^{10}\}=D\,10^9(7{,}87053-0{,}042877)$$
$$=2\,5746\,10^{-27}\,7{,}82776=\underline{2{,}032\,10^{-26}}.$$

* In the following calculations Chamber's seven-figure logarithm-table was used in the final results less figures were given to a good approximation

<u>(111)⟵(002)</u>

$$d_0 = D\{930(3^{14}-2^{14}) - 525(3^{13}-2^{13}) + 5453\, 3^{12} - 5894\, 2^{12}$$

$$-1512\, 3^{11} + 2898\, 2^{11} + 3664\, 3^{10} - 17614\, 2^{10} + 294\, 3\, 1^2\, 1^2$$

$$-1052\, 3^{12} + 294\, 3^{11} - 3024\, 3^{10}\}$$

$$= D\{98062\, 3^{10} - 56072\, 2^{10}\} = D\, 10^9(5{,}79047 - 0{,}0574177)$$

$$= 2{,}5746\, 10^{-27}\, 5{,}7330523 = \underline{1{,}449\, 10^{-26}}$$

<u>(102)⟵(101)</u>

$$d_0 = D\{930(3^{14}-2^{14}) - 525(2\, 3^{13} - 2^{13}) + 5894\, 3^{12} - 5453\, 2^{12}$$

$$-2898\, 3^{11} + 1512\, 2^{11} + 17614\, 3^{10} - 3664\, 2^{10}\}$$

$$= D\{108946\, 3^{10} - 33132\, 2^{10}\} = D\, 10^9(6{,}43316 - 0{,}30927)$$

$$= 2,\ 5746.10^{-27}.6,\ 3992 = \underline{1{,}647\ 10^{-26}}$$

<u>(211)⟵(011)</u>:

$$d_0 = D\{930(4^{14}-2^{14}) - 525(4^{13}-2^{13}) + 5453(4^{12}-2^{12}) - 1512(4^{11}$$

$$-2^{11}) + 3664(4^{10}-2^{10}) + 294\, 2^2\, 1^2\, 4^{10} - 1050\, 2\, 1\, 4^{12} + 294\, 2\, 1\, 4^{11}$$

$$-3024\, 2\, 1\, 4^{10}\}$$

$$= D\{253224\, 4^{10} - 33132\, 2^{10}\} = D.10^{11}(2{,}65524 - 0{,}00039272)$$

$$= 2{,}5746\, 10^{-25}.26485 = \underline{0{,}683\, 10^{-24}}.$$

<u>(202)⟵(002)</u>:

$$d_0 = D\{930(4^{14}-2^{14}) - 525\, 2(4^{13}-2^{13}) + 5894\, 4^{12} - 5453\, 2^{12}$$

$$-2898\, 4^{11} + 1512\, 2^{11} + 17614.4^{10} - 3664\, 2^{10}\}$$

$$= D\{275706\, 4^{10} - 41874.2^{10}\} = D\, 10^{11}(2{,}89093 - 0{,}00042877)$$

$$= 2{,}5746\, 10^{-25}\ 2{,}8905 = \underline{0{,}7428{,}10^{-24}}$$

<u>(301)⟵(101):</u>

$$d_0 = D\{930(4^{14}-2^{14})-525(4^{13}-2^{13})+5453(4^{12}-2^{12})$$
$$-1512(4^{11}-2^{11})+3664(4^{10}-2^{10})\}$$
$$=D\{328922\ 4^{10}-33132\ 2^{10}\}=D\ 10^{11}(3{,}44973-0{,}0003927)$$
$$=2{,}5746\ 10^{-25}\ 3{,}44934=\underline{0{,}8863\ 10^{-24}}$$

<u>(112)⟵(011):</u>

$$d_0=D\{930(4^{14}-2^{14})-525(2\ 4^{13}-2^{13})+(5894\ 4^{12}-5453\ 2^{12})$$
$$-2898\ 4^{11}-1512\ 2^{11})+(17614\ 4^{10}-3664\ 2^{10})+294\ 4^{10}$$
$$-1050\ 4^{12}+294\ 2\ 4^{11}-2898\ 4^{10}\}$$
$$=D\{270954\ 4^{10}-33132\ 2^{10}\}=D\ 10^{11}(2{,}84116-0{,}00039272)$$
$$=2{,}5746\ 10^{-25}\ 2{,}840768=\underline{0{,}729\ 10^{-24}}$$

<u>(103)⟵(002)</u>

$$d_0=D\{930(4^{14}-2^{14})-(1575\ 4^{13}-1050\ 2^{13})+(5873\ 4^{12}-5894\ 2^{12})$$
$$-(2898\ 4^{11}-2898\ 2^{11})+(53296\ 4^{10}-17614\ 2^{10})\}$$
$$=D\{272952\ 4^{10}-42234\ 2^{10}\}=D.10^{11}(2{,}86211-0{,}000432476)$$
$$=2{,}5746.10^{-25},2{,}86168=\underline{0{,}735\ 10^{-24}}$$

<u>(211)⟵(002)</u>

$$d_0=D\{930(4^{14}-2^{14})-(525\ 4^{13}-1050{,}2^{13})+(5453\ 4^{12}-5894\ 2^{12})$$
$$-(1512\ 4^{11}-2898\ 2^{11})+(3664\ 4^{10}-17614.2^{10})+294\ 4\ 4^{10}$$
$$-1050\ 2\ 4^{12}+294\ 2\ 4^{11}-3024.2\ 4^{10}\}$$
$$=D\{286824.4^{10}-41874\ 2^{10}\}=D\ 10^{11}(3{,}00757-0{,}00042877)$$
$$=2{,}5746\ 10^{-25}\ 3{,}00708=\underline{0{,}772\ 10^{-24}}$$

<u>(202)⟵(011)</u>

$$d_0 = D\{930(4^{14}-2^{14}) - (1050\ 4^{13} - 525\ 2^{13}) + (5894\ 4^{12} - 5453\ 2^{12})$$
$$-(2898\ 4^{11} - 1512\ 2^{11}) + (17614\ 4^{10} - 3664\ 2^{10})\}$$
$$= D\{349998\ 4^{10} - 33132\ 2^{10}\} = D\ 10^{11}(3{,}66155 - 0{,}00039272)$$
$$= 2{,}5746\ 10^{-25}\ 3{,}66115 = \underline{0{,}9423\ 10^{-24}}$$

<u>(221)⟵(011)</u>

$$d_0 = D\{930(5^{14}-2^{14}) - 525(5^{13}-2^{13}) + 5453(5^{12}-2^{12})$$
$$-1512(5^{11}-2^{11}) + 3664(5^{10}-2^{10}) + 294\ 4\ 4\ 5^{10} - 1050\ 2\ 2\ 5^{13}$$
$$+294\ 2\ 5^{11} - 3024\ 2\ 2\ 510^{10}\}$$
$$= D\{541542\ 5^{10} - 33132{,}2^{10}\} = D\ 10^{12}(5{,}2884 - 0{,}0000339)$$
$$= 2{,}5746\ 10^{-24}.5{,}28837 = \underline{13{,}59\ 10^{-24}}$$

<u>(212)⟵(002)</u>:

$$d_0 = D\{930(5^{14}-2^{14}) - 1050(5^{13}-2^{13}) + 5894(5^{12}-2^{12})$$
$$-2898(5^{11}-2^{11}) + 17614(5^{10}-2^{10}) + 294\ 4\ 5^{10} - 1050\ 2\ 5^{13}$$
$$+294\ 2\ 2\ 5^{11} - 2898\ 2\ 5^{10}\}.$$
$$= D\{546882\ 5^{10} - 41874\ 2^{10}\} = D\ 10^{12}(5{,}34064 - 0{,}0000428)$$
$$= 2{,}5746\ 10^{-24}\ 5{,}3406 = \underline{13{,}72\ 10^{-24}}$$

<u>(311)⟵(011)</u>.

$$d_0 = D\{930(5^{14}-2^{14}) - 525(5^{13}-2^{13}) + 5453(5^{12}-2^{12}) - 1512(5^{11}$$
$$-2^{11}) + 3664(5^{10}-2^{10}) + 294\ 3^2.1^2\ 5^{10} - 1050\ 3\ 1\ 5^{12}$$
$$+294\ 1\ 3\ 1\ 5^{11} + 3024\ 3\ 1\ 5^{10}\}$$
$$= D\{583432\ 5^{10} - 33132.2^{10}\} = D\ 10^{12}(5{,}7171 - 0{,}00003$$
$$= 2{,}5746.10^{-24}.5{,}7171 = \underline{14{,}7155.10^{-24}}.$$

(302)⟵(002)

$$d_0 = D\{930(5^{14}-2^{14}) - 525\,2(5^{13}-2^{13}) + (-231+672+5453).(5^{12}-2^{12}) - (-210-63+1659+1512)(5^{11}-2^{11}) + (-192-210+12840+1512+3664)(5^{10}-2^{10})\}$$

$$= D\{600474\;5^{10} - 41874.2^{10}\} = D\;10^{12}(5{,}864 - 0{,}000042879)$$

$$= 2{,}5746\;10^{-24}\;5{,}864 = \underline{15{,}2939\;10^{-24}}$$

(401)⟵(101):

$$d_0 = D\{930(5^{14}-2^{14}) - 525(5^{13}-2^{13}) + 5453(5^{12}-2^{12}) - 1512(5^{11}-2^{11}) + 3664(5^{10}-2^{10})\}$$

$$= D\{648054\;5^{10} - 33132\;2^{10}\} = D\;10^{12}(6{,}32868 - 0{,}00003)$$

$$= 2{,}5746\;10^{-24}\;6{,}32865 = \underline{16{,}268\;10^{-24}}$$

(113)⟵(002):

$$d_0 = D\{930(5^{14}-2^{14}) - 525(3.5^{13} - 2\;2^{13}) + (-231\;2^2 + 672\;2 + 5453)5^{12} - (-231\;1^2 + 672\;1 + 5453)2^{12} - (-210\;2^3 - 63\;2^2 + 1659\;2 + 1512)5^{11} + (-210\;1^3 - 63\;1^2 + 1659\;1 + 1512)2^{11} + (-192\;2^4 - 210\;2^3 + 12840\;2^2 + 1512\;2 + 3664)5^{10} - (-192\;1^4 - 210\;1^3 + 12840\;1^2 + 1512\;1 + 3664)2^{10} + 294\;1^2\;1^2.5^{10} - 1050\;1\;1\;5^{12} + 294\;3\;1\;1\;5^{11} - 1\;1\;5^{10}(-420\;2^2 + 294\;2 + 3024)\}$$

$$= D\{5^{10}\;546528 - 2^{10}\;41874\} = D\;10^{12}(5{,}3372 - 0{,}00004)$$

$$= 2{,}5746\;10^{-24}\;53368 = \underline{13{,}734\;10^{-24}}$$

(221)⟵(002)

$$d_0 = D\{930(5^{14}-2^{14}) - 525(1\;5^{13} - 2\;2^{13}) + 5453\;5^{12} - (-231\;1^2 + 672\;1 + 5453)2^{12} - 1512\;5^{11} + (-210\;1^3 - 63\;1^2 + 1659 + 1512)2^{11} + 3664\;5^{10} - (-192 - 210 + 12840 + 1512 + 3664\;2^{10}$$

$+294.2^2.2^2.5^{10}-1050\ 2\ 2.5^{12}+294.1.5\ 2.2\ 5^{10}-2\ 2\ 3024\ 5^{10}\}$

$=D\{541542.5^{10}-41874\ 2^{10}\}=D.10^{12}(5,2885-0,00004)$

$=2,5746.10^{-24}.5,2885=\underline{13,6113\ 10^{-24}}$

(212)⟵(101):

$d_0=D\{930(5^{14}-2^{14})-(2\ 525\ 5^{13}-525\ 2^{13})+(5894\ 5^{12}$

$-5453\ 2^{12})-(2898\ 5^{11}-1512.2^{11})+(17614\ 5^{10}-3664.2^{10})$

$+(294\ 2^2-2.5^2\ 1050+294\ 2\ 5.2-2898.2)5^{10}\}$

$=D\{549234\ 5^{10}-33132\ 2^{10}\}=D\ 10^{12}(5,36361-0,0000339)$

$=2,5746\ 10^{-24}.5,36358=\underline{13\ 77\ 10^{-24}}.$

(203)⟵(002):

$d_c=D\{930(5^{14}-2^{14})-525(3\ 5^{13}-2^{13})+(-231\ 2^2+672\ 2$

$+5453)5^{12}-(-231\ 1^2+672+5453)2^{12}-(-210.2^3-63\ 2^2$

$+1659.2+1512)5^{11}+(-210\ 1^3-63\ 1^2+1659+1512)2^{11}+(-192\ 2^4$

$-210\ 2^3+12840.2^2+1512\ 2+3664)5^{10}-(-192\ 1^4-210.1^3$

$+12840+1512)2^{10}\}$

$=D\{570006\ 5^{-10}.-41874.2^{10}\}=D\ 10^{12}(5,5664-0,00004)$

$=2,5746.10^{-24}.5,56636=\underline{14,584\ 10^{-24}}.$

(311)⟵(002):

$d_0=D\{930(15^{14}-2^{14})-525(1.4^{13}-2.2^{13})+5453\ 5^{12}-(-231$

$+672+5453)2^{12}-1512\ 5^{11}+(-210-63+1659+1512)2^{11}$

$+3665.5^{10}-(192-210+12840+1512+3664)2^{10}+294\ 5^{10}\ 3^2\ 1^2$

$-1040\ 5^{12}.3\ 1+294\ 1.5\ 3\ 1\ 5^{10}-3024\ 5^{10}.3\ 1\}$

$=D\{567388\ 5^{10}-41874\ 2^{10}\}=D\ 10^{12}(5,5409-0,00005)$

$=2,5746\ 10^{-24}.5,5408=\underline{14,2653\ 10^{-24}}.$

$\underline{(302) \longleftarrow (101)}$:

$$d_0 = D\{930(5^{14}-2^{14}) - 525(2.5^{13} - 2^{13}) + (-231\,1 + 672.1 + 5453)5^{12}$$

$$-5453\,2^{12} - (-210 - 63 + 1659 + 1512)5^{11} + 1512\,2^{11} + (-192$$

$$-210 + 12840 + 1512 + 3664)5^{10} - 3664\,2^{10}\}$$

$$= D\{600474\,5^{10} - 33132\,2^{10}\} = D\,10^{12}(5{,}8640 - 0{,}00003)$$

$$= 2{,}5746\,10^{-24}\,5{,}863997 = \underline{15{,}093\,10^{-24}}$$

APPENDIX VI

$H_\gamma \pi$ 18, $\gamma = \underline{4340{,}47 \text{Å}, \tilde{\nu} = 23039{,}00\ cm^{-1}}$, $\underline{\text{J in 1 Million-Volt unit}}$

$\underline{J=1}$:

1st order :—

$$\tilde{\nu} + \Delta\tilde{\nu} = 23039 + 1159{,}02 = 24198{,}02,$$

$$\therefore \lambda - \Delta\lambda = 1/(\tilde{\nu} + \Delta\tilde{\nu}) = 1/24198{,}02, \quad \log(\lambda - \Delta\lambda) = 5{,}6162202$$

$$\lambda - \Delta\lambda = 4132{,}57 \text{Å}, \qquad \Delta\lambda = \underline{207{,}90}.$$

2nd order :—

$$\tilde{\nu} + \Delta\tilde{\nu} = 23039 + 1159{,}02 - 130{,}5 = 24067{,}52,$$

$$\therefore \lambda - \Delta\lambda = 1/(\tilde{\nu} + \Delta\tilde{\nu}) = 1/24067{,}52, \quad \log(\lambda - \Delta\lambda) = \bar{5}{,}6185689$$

$$\lambda - \Delta\lambda = 4154{,}98 \text{Å}, \qquad \Delta\lambda = \underline{185{,}49}$$

3rd order :—

$$\tilde{\nu} + \Delta\tilde{\nu} = 23039 + 1159{,}02 - 130{,}5 + 29{,}30 = 24096{,}82,$$

$$\therefore \lambda - \Delta\lambda = 1/(\tilde{\nu} + \Delta\tilde{\nu}) = 1/24096{,}82, \quad \log(\lambda - \Delta\lambda) = \bar{5}{,}6180405,$$

$$\lambda - \Delta\lambda = 4149{,}7 \text{Å}, \qquad \Delta\lambda = \underline{190{,}77}.$$

4th order :—

$$\widetilde{\nu}+\Delta\widetilde{\nu}=23039+1159,02-130,5+29,30-16,26=24079,56\ ;$$

$$\therefore \lambda-\Delta\lambda=1/(\widetilde{\nu}+\Delta\widetilde{\nu})=1/24079,56\ ,\ \log(\lambda-\Delta\lambda)=\bar{5},6183514\ ,$$

$$\lambda-\Delta\lambda=4152,00\text{Å}\ ,\qquad \Delta\lambda=\underline{188,47}$$

$\underline{J=0,6}$:

1st order ,—

$$\widetilde{\nu}+\Delta\widetilde{\nu}=23039+1152,02(0,6)=23734,41\ ;$$

$$\therefore \lambda-\Delta\lambda=1/23734,41\ ;\ \log(\lambda-\Delta\lambda)=\bar{5},6246218\ ,$$

$$\lambda-\Delta\lambda=4213,3\text{Å}\ ,\qquad \Delta\lambda=\underline{127,17.}$$

2nd order :—

$$\widetilde{\nu}+\Delta\widetilde{\nu}=23039+1152,02(0,6)-130,5(0,6)^2=23687,43\ ,$$

$$\therefore \lambda-\Delta\lambda=1/23687,43\ ,\ \log(\lambda-\Delta\lambda)=\bar{5},6254819\ ;$$

$$\lambda-\Delta\lambda=4221,65\text{Å}\ ;\quad \Delta\lambda=\underline{118,32}$$

3rd order :—

$$\widetilde{\nu}+\Delta\widetilde{\nu}=23039+1152,02(0,6)-130,5(0,6)^2+29,3(0,6)^3=23693,76,$$

$$\therefore \lambda-\Delta\lambda=1/23693,76\ ,\ \log(\lambda-\Delta\lambda)=\bar{5},6253658\ ;$$

$$\lambda-\Delta\lambda=4220,52\text{Å}\ ,\quad \Delta\lambda=\underline{119,95.}$$

4th order :—

$$\widetilde{\nu}+\Delta\widetilde{\nu}=23039+1152,02(0,6)-130,5(0,6)^2+29,3(0,6)^3-16,26(0,6)^4=23693,76-2,1=23691,66\ ;$$

$$\therefore \lambda-\Delta\lambda=1/23691,66\ ;\ \log(\lambda-\Delta\lambda)=\bar{5},6254044\ ;$$

$$\lambda-\Delta\lambda=4220,9\text{Å}\ ;\quad \Delta\lambda=\underline{119,57.}$$

(Since my calculations tally perfectly with those of Gebauer and Rausch for the first three orders, except for the 2nd order when J is 1 Million-Volt/cm, it is not necessary to show calculations for the first three orders So I shall give now calculations for the 4th order only).

J=0,7 :

4th order :—

$$\tilde{\nu}+\Delta\tilde{\nu}=23039+1159{,}02(0{,}7)-130{,}5(0{,}7)^2+29{,}3(0{,}7)^3$$
$$-16{,}26(0{,}7)^4$$
$$=23039+811{,}314-63{,}945+10{,}0499-3{,}9040=23792{,}51\,,$$

$$\therefore \lambda-\Delta\lambda=1/23792{,}51\,,\ \log(\lambda-\Delta\lambda)=\bar{5}{,}6235690\,,$$

$$\lambda-\Delta\lambda=4203{,}03\text{Å}\,,\ \Delta\lambda=\underline{137{,}44}$$

J=0, 8 :

4th order :—

$$\tilde{\nu}+\Delta\tilde{\nu}=23039+1159{,}02(0{,}8)-130{,}5(0{,}8)^2+29{,}3(0{,}8)^3$$
$$-16{,}26(0{,}8)^4$$
$$=23039+927{,}216-83{,}520+15{.}0016-6{,}659096$$
$$=23891{,}038\,,$$

$$\therefore \lambda-\Delta\lambda=1/23891{,}038\,,\ \log(\lambda-\Delta\lambda)=\bar{5}{,}6217650\,,$$

$$\lambda-\Delta\lambda=4185{,}67\text{Å}\,,\ \Delta\lambda=\underline{154{,}80}$$

J=0, 9 :

4th order :

$$\tilde{\nu}+\Delta\tilde{\nu}=23039+1159{,}02(0{,}9)-130{,}6(0{,}9)^2+29{,}8(0{,}9)^3$$
$$-16{,}26(0{,}9)^4$$
$$=23039+1043{,}118-105{,}705+21{,}3597-10{,}668186$$
$$=23039+948{,}1=23987{,}1\,,$$

$$\therefore \lambda-\Delta\lambda=1/23987{,}1\,,\ \log(\lambda-\Delta\lambda)=\bar{5}{,}6200223\,;$$

$$\lambda-\Delta\lambda=4168{,}91\text{Å}\,;\ \Delta\lambda=\underline{171{,}56}$$

My best thanks are due to Prof. S. N Bose, Dacca University who helped me a great deal when the problem was taken up in 1931

WARING'S PROBLEM FOR CUBES

BY

LOOK-KENG HUA*—(*Tsing Hua*).

(Communicated by B S RAY).

The author† made use of the ternary quadratic form

$$x^2+2y^2+5z^2$$

to prove that

All large integers are sums of eight values of

$$\sigma x+\frac{x^3-x}{6}, \quad x\geq 0 \text{ integer};$$

and

All large even integers are sums of eight values of

$$\sigma x+\frac{x^3+x}{3}, \quad x\geq 0 \text{ integer}.$$

Landau‡ made use of the ternary quadratic form

$$x^2+y^2+z^2$$

to prove that

All large integers are the sums of eight values of positive cubes.

It is the purpose of this note to prove Landau's result by the quadratic form

$$x^2+2y^2+5z^2.$$

Lemma 1. If p is a prime $\equiv 2 \pmod 3$, then every integer prime to p a cubic residue (mod p^3).

Lemma 2 If a is an arbitrary integer, then there exists a γ such that

* Research fellow of the China Foundation for the Promotion of Education and Culture.

† Math. Ann (Current number)

‡ Math. Ann. Vol. 66, p. 102

$$a-\gamma^3 \equiv 0 \pmod 6,$$
$$a-\gamma^3 \not\equiv 0 \pmod 5,$$

and

$$0 \leq \gamma \leq 30$$

Lemma 3 If n is sufficiently large, there exists a prime $p \equiv 2 \pmod 3$ such that $(n, p)=1$ and

$$\sqrt[9]{\frac{n}{274}} < p \leq \sqrt[9]{\frac{n}{270}}.$$

Proof of the theorem By Lemma 3, for n sufficiently large we have a prime $p \equiv 2 \pmod 3$ such that $(n, p)=1$ and

$$270p^9 \leq n \leq 274p^9.$$

By Lemma 1, there exists an integer β such that

$$n-\beta^3=p^3M, \qquad 0<\beta<p^3.$$

Therefore

$$269p^9 \leq p^3M < 274p^9,$$
$$269p^6 \leq M < 274p^6.$$
$$p^6 \leq M-268p^6 < 6p^6$$

When n is sufficiently large, $p^6 \geqq 30^3$ By Lemma 2 $(a=M-268p^6)$,

$$6p^6 > M-268p^6 = \gamma^3+M_1$$
$$M_1 \not\equiv 0 \pmod 5, \; M_1 \equiv 0 \pmod 6$$

Therefore

$$M_1=6(A^2+2B^2+5C^2)$$

Hence for sufficiently large n

$$n=\beta^3+p^3M=p^3(268p^6+\gamma^3+M_1)$$
$$=\beta^3+p^3(268p^6+8^3\gamma^3+6(A^2+2B^2+5C^2))$$
$$=\beta^3+(p\gamma)^3+(p^3+A)^3+(p^3-A)^3+(2p^3+B)^3+(2p^3-B)^3$$
$$+(5p^3+C)^3+(5p^3-C)^3$$

Since, $M_1<6p^6$, $|A|$, $|B|$, $|C|$ are less than p^3 Hence the argument in the above expression are all positive.

Bull Cal Math Soc., Vol XXVI, Nos 3 & 4 (1934).

UEBERGEWISSE FRAGEN DER ZAHLENTHEORIE

BY

A MOSNER—(*Nurnberg*)

(*Communicated by the Secretary*)

I. Es ist

$$3^n+19^n+22^n=10^n+15^n+23^n, \quad n=2 \text{ und } n=6.$$

Wie heisst die allgemeine Losung zur Equation ?

$$A^n+B^n+C^n=D^n+E^n+F^n \quad n=2 \text{ and } n=6.$$

II. Es ist

$$2(7^n+14^n)=(-1)^n+2^n+4^n+10^n+12^n+15^n$$

$$n=1, 2, 3, 4, 5.$$

Ist die Equation

$$2(G^n+H^n)=I_1^n+I_2^n+I_3^n+I_4^n+I_5^n+I_6^n \quad n=1, 2, 3, 4, 5$$

losbar mit ganzen positiven zahlen ?

III. Es ist

$$33^2+33^3+11^5=22^3+22^2+11^3$$

$$33^6+33^5+11^6=22^6+22^5+11^9$$

Wie heisst die allgemeine Losung zur Equation ?

$$K^2+K^3+L^2=M^3+M^2+L^3$$

$$K^6+K^5+L^6=M^6+M^5+L^9.$$

CORRECTIONS

On certain definite Integrals involving Bessel Functions of the order zero by Dr S C Mitra, Vol. XXVI, No 2, pp (51-56)

P 51 6th line from the bottom

$$\textit{for} \int_0^\infty I_{n+\frac{1}{2}}(z) K_{n+\frac{1}{2}}(z) \sin 2kz \, dz,$$

$$\text{please } \textit{read} \int_0^\infty I_{n+\frac{1}{2}}(z) K_{n+\frac{1}{2}}(z) \cos 2kz \, dz$$

5th line from the bottom

$$\textit{for} \int_0^\infty \int_0^1 P_n(1-2y^2) e^{-z^2y^2} \sin 2kz \, dy dz,$$

$$\text{please } \textit{read} \int_0^\infty \int_0^1 P_n(1-2y^2) e^{-z^2y^2} \cos 2kz \, dy \, dz$$

P 52 Equation (7)

for $I_{n+\frac{1}{2}}(z)$ please *read* $I_{n+\frac{1}{2}}(t)$.

P 54. Equations (16) and (18)

for $4n+2$ please *read* $4n+1$

P 55. 1st line

for e^{-z} please *read* e^{-az}
